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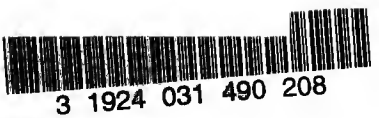
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PREFACE.

THE author's New Elementary Algebra having now, after several years' trial, proved a great success, many teachers, acquainted with the practical results accomplished by the use of that book, have been very urgent in their demands for a work on the same plan, for advanced students.

The aim of this treatise is to meet, more fully than has been done heretofore, the requirements of the highest standard of mathematical instruction in the best high schools and seminaries. To this end, great care has been taken to include all the more important parts of analysis, to treat each topic with as much conciseness as is consistent with clearness and elegance, to introduce valuable original processes, and to secure throughout an arrangement most conducive to a philosophical development of the science.

The work, therefore, is very comprehensive in its scope, ample in its range for the curriculum of the college, yet furnishing a course not too extended for the higher order of academies.

The Examples have been selected with a view to a full illustration of principles. While complicated and puzzling problems have been excluded, a proper regard has been had to such exercises as promote mental discipline, or tend to make expert practical analysts.

Each subject, as far as possible, has been made complete in itself, so that certain parts can be omitted by the student when desirable, at the option of the teacher.

The General Theory of Equations, which is often embraced in a volume by itself, is necessarily briefly treated. All the essential points, however, receive attention, and are, it is believed, rendered sufficiently intelligible. This part of the work may not always be required for the purposes of recitation, and undoubtedly will not be so used, in many cases, yet it will be found of great value for occasional reference.

Recurring Equations, Binomial Equations, Cardan's Rule for solving Cubic Equations, and other methods seldom called into use by the practical algebraist, have been put in the form of an Appendix.

The valuable works of Lefebure de Fourcy, Reynaud, Mayer and Choquet, and Todhunter, have been freely consulted; but whatever material these have furnished has been assimilated in language and style.

Many eminent teachers have kindly furnished hints and suggestions which have been of great service. But more than a general acknowledgment is due to H. B. Maglathlin, A. M. The greater part of the editorial labor of this volume has been performed by him; and he, therefore, must share largely the credit of whatever merit it may possess.

BRADFORD, MASS., Sept. 1, 1864.

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ALGEBRA.

DEFINITIONS AND NOTATION.

1. QUANTITY is anything that can be measured; as distance, time, weight, and number.

2. The UNIT of quantity is one of the same kind as the quantity, assumed as a standard, or unit of measure.

3. The MEASUREMENT of quantity is accomplished by finding the number of times the quantity contains the assumed unit of quantity.

4. MATHEMATICS is the science of quantities and their relations.

5. ALGEBRA is that branch of mathematics in which the relations of quantities are investigated, and the reasoning is abridged and generalized by means of symbols.

6. The SYMBOLS employed in Algebra are of four kinds: symbols of quantity, symbols of operation, symbols of relation, and symbols of abbreviation.

SYMBOLS OF QUANTITY.

7. The SYMBOLS OF QUANTITY generally used, are the *figures* of arithmetic and the *letters* of the alphabet.

The *figures* are used to represent known quantities and determined values, and the *letters*, any quantity whatever, known or unknown.

8. KNOWN QUANTITIES, or those whose values are given, when not expressed by figures, are represented by the first letters of the alphabet, as a, b, c.

9. UNKNOWN QUANTITIES, or those whose values are not given, are represented by the last letters of the alphabet, as x, y, z.

10. ZERO, or the absence of quantity, or that which is less than any assignable quantity, is represented by the symbol 0.

11. INFINITY, or that which is greater than any assignable quantity, is represented by the symbol ∞.

12. Quantities occupying similar relations in different operations are often represented by the same letter, distinguished by different *accents*, as a', a'', a''', read, a prime, a second, a third, etc.; or by different *subscript figures*, as a_1, a_2, a_3, a_4, read, a sub one, a sub two, a sub three, etc.

SYMBOLS OF OPERATION.

13. The SYMBOLS OF OPERATION are certain *signs* or *characters* used to indicate algebraic operations.

14. The SIGN OF ADDITION, $+$, is called *plus*. Thus, $a + b$, read a plus b, indicates that the quantity b is to be added to the quantity a.

15. The SIGN OF SUBTRACTION, $-$, is called *minus*. Thus, $a - b$, read a minus b, indicates that the quantity b is to be subtracted from the quantity a.

The sign $\sim$ may be written between two quantities, when it is not known which of them is the greater. Thus, $a \sim b$ indicates the difference of the two quantities a and b.

16. The SIGN OF MULTIPLICATION, $\times$, is read *into*, or *multiplied by*. Thus, $a \times b$ indicates that the quantity a is multiplied by the quantity b.

A simple point (.) is sometimes used in the place of

the sign $\times$. The sign of multiplication is, however, usually omitted, except between two arithmetical figures separated by no other sign, and multiplication is therefore indicated by the absence of any sign. Thus, $2\,a\,b$ indicates the same as $2 \times a \times b$, or $2 \,.\, a \,.\, b$.

17. The quantities multiplied are called *factors*, and the result of the multiplication is called the *product*.

18. The SIGN OF DIVISION, $\div$, is read *divided by*. Thus, $a \div b$ indicates that the quantity a is divided by the quantity b.

Division is otherwise often indicated by writing the dividend above and the divisor below a short horizontal line. Thus, $\frac{a}{b}$ indicates the same as $a \div b$. Also, the sign of division may be replaced in an operation by a straight or curved line. Thus, $a \,|\, \underline{b}$, or $b\,)\,a$, indicates the same as $a \div b$.

19. The SIGN OF INVOLUTION, or EXPONENTIAL SIGN, is a figure or letter written at the right and above a quantity, to indicate the number of times the quantity is taken as a factor. Thus, in x^3, the ^{3} indicates that x is to be taken three times as a factor; and x^3 is equivalent to $x\,x\,x$.

The product obtained by taking a factor one or more times is called a *power*. Thus,

a^2, read a square, indicates the *second* power of a;
a^3, read a cube, indicates the *third* power of a;
a^4, read a fourth, indicates the *fourth* power of a;
a^n, read a nth, indicates the nth power of a.

The figures or letters used to indicate powers are called *exponents*; and when no exponent is written, the *first* power is understood. Thus, a is equivalent to a^1.

If the minus sign is prefixed to an exponent, it indicates that the quantity is to be used as a divisor. Thus, $a^2\,b^{-1}\,c^{-3}$ is the same as $\frac{a^2}{b\,c^3}$, and x^{-2} is the same as $\frac{1}{x^2}$.

The *root* of a quantity is one of its equal factors. Thus the root of a^2, a^3, a^4, is a.

20. The SIGN OF EVOLUTION, or RADICAL SIGN, $\surd$, when prefixed to a quantity, indicates that some root of the quantity is to be extracted. Thus,

$\sqrt[2]{a}$ indicates the *square* or *second root* of a;
$\sqrt[3]{a}$ indicates the *cube* or *third root* of a;
$\sqrt[4]{a}$ indicates the *fourth root* of a; and so on.

The *index* of the root is the figure or letter written over the radical. Thus, ^{2} is the index of the square root, ^{3} of the cube root, and so on.

When the radical sign has no index written over it, the index ^{2} is understood. Thus, $\surd a$ is the same as $\sqrt[2]{a}$.

A *fractional exponent* is also used to indicate a root. Thus, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, indicate, severally, the square, cube, and fourth root of a.

In fractional exponents the numerator denotes a *power*, and the denominator a *root*, thus, $b^{\frac{5}{4}}$ indicates the fifth power of the fourth root of b, or the fourth root of the fifth power of b.

SYMBOLS OF RELATION.

21. The SYMBOLS OF RELATION are signs used to indicate the relative magnitude of quantities.

22. The SIGN OF EQUALITY, $=$, read *equals*, or *equal to*, indicates that the quantities between which it is written are equal. Thus, $x = y$, indicates that the quantity x equals the quantity y.

An expression in which quantities are connected by the sign $=$ is an *equation*, as $x + 4 = 2x - 1$, read x plus 4 equals $2x$ minus 1.

23. The SIGN OF RATIO (:), read *to*, indicates that the two quantities between which it is placed are taken as the terms of a ratio. Thus, $a : b$ indicates the ratio of

the quantity a to the quantity b; and is read the ratio of a to b.

An *equality* of *ratios*, or *a proportion*, is expressed by writing the sign $=$, or the sign $(::)$, between equal ratios. Thus, $30 : 6 = 25 : 5$ is read the ratio of 30 to 6 equals the ratio of 25 to 5, or 30 is to 6 as 25 to 5.

24. The SIGN OF INEQUALITY, $>$ or $<$, read *is greater than*, or *is less than*, when placed between two quantities, indicates that the quantity toward which the opening of the sign turns is the greater. Thus, $x > y$ is read x is greater than y, $x - 6 < y$ is read x minus 6 is less than y.

25. The SIGN OF VARIATION, $\propto$, read *varies as*, indicates that the two quantities between which it is placed increase or diminish together. Thus, $a \propto \frac{c}{d}$, is read a varies as $\frac{c}{d}$.

SYMBOLS OF ABBREVIATION.

26. The SIGNS OF DEDUCTION ($\therefore$ and $\because$) stand the one for *therefore* or *hence*, and the other for *since* or *because*.

27. The SIGNS OF AGGREGATION, the *vinculum* ——, the *bar* |, the *parenthesis* ( ), the *brackets* [ ], and the *braces* $\left\{ \right\}$, indicate that the quantities connected, or inclosed, by them are to be taken as one quantity, and to be operated upon in the same way. Thus,

$\overline{a + b} \times x$, $\begin{matrix} + a \\ + b \end{matrix} \Big| x$, $(a + b)\, x$, $[a + b]\, x$, $\left\{ a + b \right\} x$,

each indicate that the quantity $a + b$ is to be multiplied by x.

28. The SIGN OF CONTINUATION (. . . . .) stands for *and so on*, or *continued by the same law*. Thus,

$a,\ a + b,\ a + 2\,b,\ a + 3\,b,$ . . . . . is read
$a,\ a + b,\ a + 2\,b,\ a + 3\,b,$ and so on.

ALGEBRAIC EXPRESSIONS.

29. An ALGEBRAIC EXPRESSION is a quantity written in algebraic language.

30. A COEFFICIENT of a quantity is a figure or letter prefixed to it to show how many times the quantity is to be taken. Thus, in $4a$, 4 is the coefficient of a, and indicates that a is taken four times; and in bxy, b may be regarded as the coefficient of xy, or bx as the coefficient of y.

When no coefficient of a quantity is written, 1 is understood to be the coefficient. Thus, a is the same as $1a$, and xy is the same as $1xy$.

31. The TERMS of an algebraic expression are its parts connected by the signs + or −. Thus,

a and b are the terms of the expression $a + b$;

$2a$, b^2, and $-2ac$, of the expression $2a + b^2 - 2ac$.

32. The DEGREE of a term is the number of literal factors which it contains. Thus,

$2a$ is of the *first* degree, since it contains but *one* literal factor.

ab is of the *second* degree, since it contains but *two* literal factors.

$3ab^2$ is of the *third* degree, since it contains but *three* literal factors.

The degree of any term is determined by adding the exponents of its several letters. Thus,

ab^2c^3 is of the *sixth* degree.

33. POSITIVE TERMS are those having the *plus* sign; as,

$$+2a, \text{ or } +ab^2.$$

When a term has no sign written, it is understood to be positive. Thus, a is the same as $+a$.

NEGATIVE TERMS are those having the *minus* sign; as, $-3a$, or $-bc$. This sign should never be omitted.

34. In a positive term the coefficient indicates how many times the quantity is taken *additively*, and in a negative term it indicates how many times the quantity is taken *subtractively*. Thus, $+2x$ is the same as $+x$ $+x$, and $-2x$ is the same as $-x$ $-x$.

The signs $+$ and $-$ are not only used to indicate operations (Art. 14, 15), but the *nature* of the quantities to which they are prefixed; that is,

A quantity affected by the plus sign is *positive*, and a quantity affected by the minus sign is *negative*.

A positive quantity is *additive*, while a negative quantity is *subtractive*, and necessarily sustains opposite relations; consequently, equal terms affected by unlike signs, in an expression, cancel. Thus, $ab + cd - cd$ is the same as ab.

Quantities independently additive or subtractive, may be regarded as added to, or subtracted from, 0, the neutral point, or starting-point, of all positive and negative quantities.

35. Similar or Like Terms are those containing the same letters, affected by the same exponents.

Thus, $2xy$ and $-7xy$ are similar terms;
also, $3a^2b^3$ and $9a^2b^3$ are similar terms.

36. Dissimilar or Unlike Terms are those containing different letters or exponents.

Thus, ab and ad are dissimilar terms;
also, bx^2y and bxy^2 are dissimilar terms.

37. A Monomial or Simple Quantity is an algebraic expression consisting of only one term; as, $5a$, $7ab$, or $3b^2c$.

38. A Polynomial or Compound Quantity is an algebraic expression consisting of more than one term; as,

$$a + b, \text{ or } 3a^2 + b - 5b^3.$$

A polynomial is sometimes called a *multinomial*.

39. A BINOMIAL is a polynomial of two terms; as,

$$a - b,\ 2a + b^2,\ \text{or}\ 3ac^2 - b.$$

A binomial whose second term is negative, as $a - b$, is sometimes called a *residual*.

40. A TRINOMIAL is a polynomial of three terms; as,

$$a + b + c,\ \text{or}\ ab + c^2 - b^3.$$

41. HOMOGENEOUS TERMS are those of the same degree. Thus, the terms a^2, $3bc$, $-4x^2$ are homogeneous.

42. A POLYNOMIAL is HOMOGENEOUS when all its terms are homogeneous. Thus, the polynomial $a^3 + abc - b^3$ is homogeneous.

43. The RECIPROCAL of a quantity is 1 divided by that quantity. Thus, the reciprocal of a is $\frac{1}{a}$, and of $x + y$ is $\frac{1}{x+y}$.

The *reciprocal of a fraction* is that fraction inverted. Thus, $\frac{n}{m}$ is the reciprocal of $\frac{m}{n}$.

INTERPRETATION OF ALGEBRAIC EXPRESSIONS.

44. The INTERPRETATION of an algebraic expression consists in rendering it into Arithmetic, by means of the numerical values assigned to its letters.

45. The NUMERICAL VALUE of an algebraic expression is the result obtained by substituting for its letters their numerical values, and then performing the operations indicated.

Thus, the numerical value of

$$4a + 3bc - d,$$

when $a = 4$, $b = 3$, $c = 5$, and $d = 2$,

is $$4 \times 4 + 3 \times 3 \times 5 - 2 = 59.$$

AXIOMS.

46. An Axiom is a self-evident truth.

Algebraic operations are based upon definitions and the following axioms: —

1. If the same quantity, or equal quantities, be *added* to equal quantities, the *sums* will be equal.

2. If the same quantity, or equal quantities, be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be *multiplied* by the same quantity, or equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same quantity, or equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added* to and *subtracted* from another, the value of the latter will not be changed.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be changed.

7. Quantities which are equal to the same quantity are equal to each other.

8. Like powers and like roots, of equal quantities, are equal.

9. The whole of a quantity is equal to the sum of all its parts.

ALGEBRAIC PROCESSES.

47. The Processes of Algebra, in general, are only those of Arithmetic extended, or rendered more comprehensive, by the aid of symbols. (Art. 7.)

Exercises on preceding Definitions and Principles.

48. Put in the form of algebraic expressions: —

1. Five times b, added to two times a. Ans. $2a + 5b$.
2. Two times x, minus y second power.
3. The difference of x and y. Ans. $x - y$.
4. ab multiplied by c square and by d cube.

5. $x + y$ divided by $a - b$.

6. The difference of a square and b square.

7. One third of x, increased by two, is equal to three y diminished by eleven. Ans. $\frac{x}{3} + 2 = 3y - 11$.

8. The reciprocal of two fifths of a, plus the square of a minus the square of b, is equal to the square root of c square plus d square.

9. The ratio of five a divided by b to d divided by c square equals the ratio of x square y cube to y square z fourth.

10. The cube root of the reciprocal of a plus b divided by $c - d$, is greater than the product of the square of $a - d$ into the third power of $a + b$, less one fourth of three x.

49. Interpret and give the numerical values of the following expressions.

When $a = 6$, $b = 5$, $c = 4$, and $d = 1$.

1. $a^2 + 2ab - c + d$.

Ans. $36 + 60 - 4 + 1 = 93$.

2. $2a^3 - 3a^2b + c^3$.

Ans. $432 - 540 + 64 = -44$.

3. $2a^2 + 3bc - 5$.

4. $a^2 \times (a + b) - 2abc$.

5. $5a^2b - 10ab^2 + 27c$.

6. $7a^2 + (b - a) \times (a - c)$.

When $a = 4$, $b = 2$, $c = 3$, and $d = 1$.

7. $15a - 7(b + c) - d$.

8. $15a^2 - 7(b^2 + c^2) + d^2$.

9. $\frac{a}{b} + \frac{b}{c} + \frac{c}{d}$.

10. $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{d^2}$.

11. $\frac{4}{3a - 3c} + \frac{8}{3}$.

12. $\frac{25a - 30c - d}{b + c}$.

When $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = \frac{1}{5}$, $d = \frac{2}{7}$, and $x = 2$.

13. $(2a + 3b + 5c)(8a + 3b - 5c)(2a - 3b + 15c)$.

14. $x^3 - \left(\frac{1}{a} + \frac{1}{b}\right)x^2 + \left(\frac{1}{b} - \frac{1}{a}\right)x + \frac{2}{b^2}$. Ans. 8.

15. $x^4 - (2a + 3b)\, x^3 + (3a - 2b)\, x^2 - cx + bc$.

When $a = b$, and $b = \frac{1}{3}$.

16. $\dfrac{5a + b - [3a - (2a - b)]}{a}$ Ans. 4.

17. $\dfrac{13a + 3b - \left\{7\,(a + b) - [3a + 8\,(a - b)]\right\}}{2a + 3b}$

When $b = 3$, $c = 4$, $d = 6$, and $e = 2$.

18. $\sqrt{(27\,b)} - \sqrt[3]{(2\,c)} + \sqrt{(2\,e)}$. Ans. 9.

19. $\sqrt{(3\,b\,c)} + \sqrt[3]{(9\,c\,d)} - \sqrt[3]{(2e^2)}$.

When $a = 16$, $b = 10$, $x = 5$, and $y = 1$.

20. $(b - x)\,(\sqrt{a} + b) + \sqrt{\left\{(a - b)\,(x + y)\right\}}$.

Ans. 76.

ADDITION.

50. Addition, in Algebra, is the process of collecting two or more quantities into one equivalent expression, called the Sum.

In algebraic addition there may be distinguished three cases, depending upon the similarity and signs of the terms.

CASE I.

51. When the terms are similar, and have the same sign.

1. Let it be required to find the sum of $3\,ab$, $7\,ab$, and $8\,ab$.

OPERATION.
$3\,ab$
$7\,ab$
$8\,ab$
$18\,ab$

Since ab, whatever it represents, is taken 3 times in the first term, 7 times in the second, and 8 times in the third, or in all 18 times, the sum of the terms is $18\,ab$.

2. Let it be required to find the sum of $-7b$, $-5b$, and $-6b$.

OPERATION.	
$-7b$	Since b, whatever it represents, is taken -7 times in the first term, -5
$-5b$	times in the second, and -6 times in
$-6b$	the third, or in all -18 times, the sum
$-18b$	of the terms is $-18b$.

CASE II.

52. When the terms are similar, and have different signs.

1. Let it be required to find the sum of $7ax$, $-4ax$, $-5ax$, and $17ax$.

OPERATION.	
$7ax$	The sum of $7ax$ and $17ax$ is $24ax$,
$-4ax$	and of $-4ax$ and $-5ax$ is $-9ax$.
$-5ax$	But since equal terms, affected by un-
$17ax$	like signs, cancel (Art. 34), $-9ax$ can-
$15ax$	cels $+9ax$ in the quantity $+24ax$,
	and leaves $15ax$ for the sum required.

2. Let it be required to find the sum of $8xy$, $3xy$, $-6xy$, and $-10xy$.

OPERATION.	
$8xy$	The sum of $-10xy$ and $-6xy$ is
$3xy$	$-16xy$, and of $3xy$ and $8xy$ is $11xy$.
$-6xy$	But $+11xy$ cancels $-11xy$ in the
$-10xy$	quantity $-16xy$, and leaves $-5xy$ for
$-5xy$	the sum required.

Hence,

The ALGEBRAIC SUM *of two quantities, the one positive and the other negative, is numerically their* DIFFERENCE, *with the sign of the greater prefixed.*

It follows from the two preceding cases, that

To ADD *similar terms is to* REDUCE *them, by finding their numerical sum or difference, to a single equivalent term.*

CASE III.

53. When the terms are dissimilar, or some similar, and others dissimilar.

1. Let it be required to find the sum of $3ax$, $4mn$, $-6y^2$, and $7xy$.

OPERATION.

$$3ax + 4mn - 6y^2 + 7xy.$$

The given terms being dissimilar, the quantities have no common unit, consequently, we can only *indicate* their addition, by uniting them by their respective signs, and have $3ax + 4mn - 6y^2 + 7xy$ for the sum required.

2. Let it be required to find the sum of $6a - 7x$, $7a - 4x$, and $-a + 6x + 4xy$.

OPERATION.

$$\begin{array}{r} 6a - 7x \\ 7a - 4x \\ -a + 6x + 4xy \\ \hline 12a - 5x + 4xy \end{array}$$

Finding the sum of the similar terms, as in Art 52, we obtain $12a - 5x$, and there being no term similar to $+4xy$, it is united by its proper sign to those terms, which gives $12a - 5x + 4xy$ for the sum required.

Hence,

The ALGEBRAIC SUM *of dissimilar terms can only be* INDICATED, *by uniting them by their respective signs.*

It is also evidently immaterial in what order the terms are united, provided each has its proper sign. Thus, $-b + a$ is the same as $a - b$.

54. From the preceding principles and illustrations is deduced the following

General Rule.

Find the sum of the similar terms, and to the result obtained unite the dissimilar terms, if any, by their proper signs.

Examples.

(1.)	(2.)	(3.)	(4.)	(5.)
$7a$	$-6m$	$4ax$	$13n$	$7a - mp^2$
$3a$	m	$-3ax$	n	$a + 6mp^2$
a	$-11m$	ax	$-20n$	$-11a - 3mp^2$
$5a$	$5m$	$-7ax$	$6n$	$8a + 11mp^2$
$11a$	$-m$	ax	$8n$	$-9a - 7mp^2$
a	$20m$	$12ax$	$-n$	$18a - 15mp^2$
$28a$	$8m$			$14a - 9mp^2$

6. Add together $a + x$ and $y - c$.

Ans. $a - c + x + y$.

7. Add $3a + b - 10$, $c - d - a$, $-4c + 2a - 3b - 7$, and $4x^2 + 5 - 18m$ together.

Ans. $4a - 2b - 12 - 3c - d + 4x^2 - 18m$.

8. Add $7a - 5y^3$, $8\sqrt{x} + 2a$, $5y^3 - \sqrt{x}$, and $-9a + 7\sqrt{x}$ together. Ans. $14\sqrt{x}$.

9. Add $4mn + 3ab - 4c$, $3x - 4ab + 2mn$, and $3m^2 - 4p$ together.

Ans. $6mn - ab - 4c + 3x + 3m^2 - 4p$.

10. Find the sum of $3a^2 + 2ab + 4b^2$, $5a^2 - 8ab + b^2$, $-a^2 + 5ab - b^2$, $18a^2 - 20ab - 19b^2$, and $14a^2 - 3ab + 20b^2$.

11. Find the sum of $4x^3 - 5a^3 - 5ax^2 + 6a^2x$, $6a^3 + 3x^3 + 4ax^2 + 2a^2x$, $-17x^3 + 19ax^2 - 15a^2x$, $13ax^2 - 27a^2x + 18a^3$, $3a^2x - 20a^3 + 12x^3$, and $31a^2x - 2x^3 - 31ax^2 - 7x^3$. Ans. $-7x^3 - a^3$.

12. Find the sum of $3ab + 3(a + b)$, $-ab + 2(a + b)$, $7ab - 4(a + b)$, and $-2ab + 6(a + b)$.

Ans. $7ab + 7(a + b)$.

13. Find the sum of $7\sqrt{y} - 4(a + b)$, $6\sqrt{y} + 2(a + b)$, $2\sqrt{y} + (a + b)$, and $\sqrt{y} - 3(a + b)$.

14. Find the sum of $3(c + d)x^3 + 4y - 2\sqrt{y}$, $6x^2y - 2ax + 12$, $y + \sqrt{y} - 5ax$, $3ax - x^2y + (c + d)x^3$, and $x^2y + 2y - 14$.

Ans. $4(c + d)x^3 + 7y - \sqrt{y} + 6x^2y - 4ax - 2$.

55. When dissimilar terms have a *common factor* (Art. 17), the addition may be expressed, by prefixing the sum of the coefficients, inclosed in a parenthesis, to the common factor.

15. Required the sum of ay^3, by^3, and cy^3.

OPERATION.

$$\begin{array}{r} ay^3 \\ by^3 \\ cy^3 \\ \hline (a + b + c)y^3 \end{array}$$

Since a times, b times, and c times y^3 will equal the sum of $a + b + c$ into y^3, $(a + b + c)y^3$ will indicate the sum required.

16. Add ax^2, bx^2, $2cx^2$, and $3dx^2$.

Ans. $(a + b + 2c + 3d)x^2$.

17. Add $amx + 2dy$, $2cx - 3dy$, and $3dx + 5y$.

Ans. $(am + 2c + 3d)x + (5 - d)y$.

18. Add $4ax - 5my$, $3dx + 7ny$, and $7mx + 4my$.

19. Add $3nz - 5x$, $4mz + nx$, and $5az - 4mx$.

Ans. $(3n + 4m + 5a)z + (n - 5 - 4m)x$.

20. Add $ax + 5y$, $6ax - y$, and $2bx + 2y$.

Ans. $(7a + 2b)x + 6y$.

21. Add $(2a + c)(m - \sqrt{m})$, $b(m - \sqrt{m})$, and $-3c(m - \sqrt{m})$. Ans. $(2a + b - 2c)(m - \sqrt{m})$.

22. Add $ax + dy^2$, $by - dx$, and $-by^2 + my$.

Ans. $(a - d)x + (d - b)y^2 + (b + m)y$.

SUBTRACTION.

56. SUBTRACTION, in Algebra, is the process of finding the difference between two algebraic quantities.

The *Subtrahend* is the quantity subtracted.

The *Minuend* is the quantity from which it is subtracted.

The *Difference*, or *Remainder*, is the quantity left after the subtraction is performed.

In algebraic subtraction there may be distinguished two cases.

CASE I.

57. When the quantities are positive.

1. Let it be required to take $3a$ from $8a$.

OPERATION.

$8a - 3a = 5a$

Since 8 times any quantity less 3 times the quantity equals 5 times the quantity, $8a$ less $3a$ equals $5a$.

2. Let it be required to take $8b$ from $5b$.

OPERATION.

$5b - 8b = -3b$

We cannot, *numerically* take 8 times any quantity from 5 times the same quantity. If we take $5b$ from $5b$, nothing will remain; there is yet, however, a quantity, $3b$, *to be subtracted*, with nothing to take it from, which, from its nature (Art. 34), we can indicate by $-3b$.

3. Let it be required to take $b + c$ from a.

OPERATION.

$a - (b + c) = a - b - c$

If b is taken from a, we have $a - b$. But a is to be diminished by c, as well as b, consequently the true remainder will be $a - b$ diminished by c, or $a - b - c$.

CASE II.

58. When the quantity to be subtracted is partly, or wholly, negative.

1. Let it be required to take $b - c$ from a.

OPERATION.

$$a - (b - c) = a - b + c$$

If we take b from a, we have $a - b$. But, in doing this, we take away c too much, consequently the true difference will be $a - b$ increased by c, or $a - b + c$.

Hence,

The ALGEBRAIC DIFFERENCE *between two quantities may be numerically greater than either quantity.*

The term *difference*, in algebra, it will be observed, has a much more general signification than in arithmetic; the same is also equally true of the term *sum* (Art. 53).

59. It follows, from the two preceding cases, that

Subtracting a positive quantity is the same as adding an equal negative quantity, and subtracting a negative quantity is the same as adding an equal positive quantity.

Hence, the following

GENERAL RULE.

Conceive the signs of all the terms of the subtrahend to be changed, from + to —, or from — to +, and then proceed as in addition.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$27\,a$	$7\,x$	$-16\,x$	$-8\,d$
$13\,a$	$14\,x$	$-5\,x$	$-11\,d$
$14\,a$	$-7\,x$	$-11\,x$	$3\,d$

(5.)

$$\begin{array}{r} ab + cd - ax \\ 4ab - 3cd + 4ax \\ \hline -3ab + 4cd - 5ax \end{array}$$

(6.)

$$\begin{array}{r} 7x + 5y - 3a \\ x - 7y + 5a \\ \hline 6x + 12y - 8a \end{array}$$

(7.)

$$\begin{array}{r} 7abc - 11x + 5y - 48 \\ 11abc + 3x + 7y + 100 \\ \hline -4abc - 14x - 2y - 148 \end{array}$$

(8.)

$$\begin{array}{r} 5\sqrt{a} - 3y^2 + 7a^{\frac{1}{2}} - 1 \\ 3\sqrt{a} + y^2 - 5a^{\frac{1}{2}} + 7 \\ \hline 2\sqrt{a} - 4y^2 + 12a^{\frac{1}{2}} - 8 \end{array}$$

9. From $31x^2 - 3y^2 + ab$ take $17x^2 + 5y^2 - 4ab + 7$.

10. Take $a - b$ from $a + b$. Ans. $2b$.

11. Take $6a - 3b - 5c$ from $6a + 3b - 5c + 1$.

12. From $5xy + 4a^2 - 6b^2$ take $3xy - 9a^2 - b^2 + 4$.

13. From $11a - 7b + c$ take $a + 7b - 3c + 11$.
Ans. $10a - 14b + 4c - 11$.

14. From $m^2 + 3n^3$ take $-4m^2 - 6n^3 + 71x$.
Ans. $5m^2 + 9n^3 - 71x$.

15. From $31a - 15x - 7$ take $2a - 25x + y^2$.
Ans. $29a + 10x - y^2 - 7$.

16. From $a + b$ take $2a - 2b$ and $-a + b$.
Ans. $2b$.

17. From $a - b - c$ take $-a + b + c$ and $a - b + c$.
Ans. $a - b - 3c$.

60. It is sometimes sufficient to *indicate* the subtraction of a polynomial, by inclosing it in a parenthesis, and prefixing the sign —. Thus,

$$5a^2 - (a^2 + b^2 - c)$$

indicates that the entire quantity $a^2 + b^2 - c$ is to be taken from $5a^2$.

Performing the operation indicated, by the rule, Art. 59, we have

$$5a^2 - a^2 - b^2 + c, \text{ or } 4a^2 - b^2 + c.$$

The first of these expressions may be transformed to its previous equivalent form, by changing the signs of the last three terms and inclosing them in a parenthesis, with the sign — prefixed. Hence,

When an expression within a parenthesis is preceded by the sign —, the parenthesis may be removed, if the sign of every inclosed term be changed; and

Any number of terms in an expression may be inclosed by a parenthesis and the sign — prefixed, if the sign of every term inclosed be changed.

18. Indicate the subtraction of $6a + b^2$ from $-7a - b$.

Ans. $-7a - b - (6a + b^2)$.

19. Indicate the subtraction of $b - c - d - e$ from a.

20. Reduce the expression $3a + b - (a - b)$ to its simplest form. Ans. $2a + 2b$.

21. Reduce the expression $2a^3 - 3a^2b + 4ab^2 - (a^3 + b^3 + ab^2)$ to its simplest form.

Ans. $a^3 - 3a^2b + 3ab^2 - b^3$.

22. Remove the parenthesis from the expression $6a - 9x - (4a - 5x)$.

23. Remove the parentheses from $4a - 5x - (a - 4x) + (x - 8a)$. Ans. $-5a$.

24. Place in a parenthesis, with the negative sign prefixed, the last three terms of the expression $2a^3 - 3a^2b + 4ab^2 - a^3 - b^3 - ab^2$.

Ans. $2a^3 - 3a^2b + 4ab^2 - (a^3 + b^3 + ab^2)$.

25. To what is $5a - 4b + 3c + (-3a + 2b - c)$ equal? Ans. $2a - 2b + 2c$.

26. To what is $6x^2 + 2y^2 - (3x^2 + y^2) - [2x^2 + 4y^2 - (4x^2 - y^2)]$ equal? Ans. $5x^2 - 4y^2$.

27. Simplify $1 - [2 - (7 - \overline{x - 4})] + 2 - [3 + (4 - \overline{x - 5})]$. Ans. 0.

61. When dissimilar terms have a *common* factor, the

subtraction may be indicated by prefixing the difference of their coefficients, inclosed in a parenthesis, to the common factor.

28. From $ay - b$ take $dy - c$.

Ans. $(a - d)\,y - b + c$.

29. From $ax^3 + dy^2$ take $bx^3 - cy^2$.

Ans. $(a - b)\,x^3 + (d + c)\,y^2$.

30. From $ax - bx + cx$ take $x + ax - bx$.

Ans. $(c - 1)\,x$.

31. From $ax^3 - bx^2 + cx - d$ take $bx^2 + ex - 2d$.

Ans. $ax^3 - 2bx^2 + (c - e)\,x + d$.

32. Take $3bxy - (5 + c)\,z^2 - (3a - 2b)\,(x + y)^{\frac{1}{2}}$ from $(4a - 5b)\sqrt{x + y} + (a - b)\,xy - cz^2$.

Ans. $(7a - 7b)\sqrt{x + y} + (a - 4b)\,xy + 5z^2$.

MULTIPLICATION.

62. MULTIPLICATION, in Algebra, is the process of taking one quantity as many times as there are units in another quantity.

The *Multiplicand* is the quantity to be multiplied, or taken.

The *Multiplier* is the quantity by which we multiply.

The *Product* is the result of the operation.

The multiplicand and multiplier are often called *factors*.

63. *The* PRODUCT *of factors is the same, in whatever order they are taken.*

For, the product contains one factor as many times as there are units in the other. Thus, the product of $a \times b$, or $b \times a$, will be ab units, since b taken a times is the same as a taken b times. Let $a = 4$ and $b = 3$, we have 4×3, or 3×4, equal to 12.

In like manner, it may be shown that the principle holds when there are any number of factors.

64. To determine the law of the COEFFICIENTS of the product.

Let it be required to find the product of $7a$ by $2b$.

Since the factors can be taken in any order (Art. 63), $7a \times 2b$ is the same as $7 \times 2 \times a \times b$; then, 2 times 7 $= 14$; b times $a = ab$; and 14 times $ab = 14ab$. Hence,

The COEFFICIENT *of the product is equal to the product of the coefficients of the factors.*

65. To determine the law of the EXPONENTS of the product.

Let it be required to find the product of $2a^3$ by a^2.

Since the exponent of a quantity indicates the number of times the quantity is taken as a factor (Art. 19), $2a^3$ is the same as $2aaa$, and a^2 is the same as aa; then, aa times $2aaa = 2aaaaa = 2a^5$. Hence,

The EXPONENT *of a letter in the product is equal to the sum of its exponents in the factors.*

66. To determine the law of the SIGNS of the product.

1. Let it be required to find the product of $+a$ by $+b$.

Since a is to be taken as many times as there are units in b, and as the *sum* of any number of positive quantities is positive, the product, ab, must be positive, or $+ab$.

2. Let it be required to find the product of $-a$ by $+b$.

Since $-a$ is to be taken as many times as there are units in b, and as the *sum* of any number of negative quantities is negative, the product must be negative, or $-ab$.

3. Let it be required to find the product of $+a$ by $-b$.

The negative multiplier, $-b$, indicates that a is to be

taken *subtractively* (Art. 34) as many times as there are units in b; and since a positive quantity becomes negative by subtraction (Art. 59), the product must be negative, or $-ab$.

4. Let it be required to find the product of $-a$ by $-b$.

Here the negative multiplier, $-b$, indicates that $-a$ is to be taken subtractively as many times as there are units in b, and since a negative quantity becomes positive by subtraction, the product must be positive, or ab.

From these results it follows,

$+$ *multiplied by* $+$, *and* $-$ *multiplied by* $-$, *produce* $+$,
$-$ *multiplied by* $+$, *and* $+$ *multiplied by* $-$, *produce* $-$.

Or, as may be enunciated for the sake of brevity, with regard to the product of any two terms,

Like signs produce $+$, *and unlike signs produce* $-$.

CASE I.

67. When both factors are monomials.

From the preceding examples is derived the following

RULE.

Multiply the numerical coefficients together; annex to the result the letters of both monomials, giving to each letter an exponent equal to the sum of its exponents in the two factors.

Make the product positive when the two factors have like signs, and negative when they have different signs.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$3ab$	$17abc$	a^3b^2c	$2x^m$
$2ac$	$-8abc$	a^2bd	$4x^n$
$6a^2bc$	$-136a^2b^2c^2$	a^5b^2cd	$8x^{m+n}$

5. Multiply $15\,m^5 n^3$ by $3\,m\,n$.

6. Multiply $3\,a^m b^n$ by $2\,a^m b^3$. Ans. $6\,a^{2m} b^{n+3}$.

7. Multiply $4\,x^m y^n$ by $-x^n y^n z^5$.

Ans. $-4\,x^{m+n} y^{2n} z^5$.

8. Multiply $17\,a^4 c^2$ by $4\,a\,a\,c\,c$. Ans. $68\,a^6 c^4$.

9. Multiply $3\,a^m b^n$ by $-4\,a^n d^n$.

Ans. $-12\,a^{2m} b^n d^n$.

10. Multiply $7\,a^m$ by $3\,a^m$. Ans. $21\,a^{2m}$.

11. Multiply $11\,n^2$ by $-5\,n^6$.

12. Multiply $4\,a^6$ by $-3\,a\,b\,y^2$. Ans. $-12\,a^7 b y^2$.

13. Multiply $7\,m^n$ by $3m^n$. Ans. $21m^{2n}$.

14. Multiply $6\,a\,b^2$ by $a^3 b$. Ans. $6a^4 b^3$.

15. Multiply $12\,a^2 x$ by $-2\,x^2 y$. Ans. $-24\,a^2 x^3 y$.

16. Multiply $6\sqrt{a\,x}$ by $4\,b$. Ans. $24\,b\sqrt{a\,x}$.

17. Multiply $8\,a\,x^2 y^2$ by $-2\,a^3 x^4 y^3$, and the product by $-4\,a^2 x\,y^2$. Ans. $64\,a^6 x^7 y^8$.

18. Find the product of $2\,m\,(a-b)^2$ by $m\,(a-b)$.

Ans. $2\,m^2\,(a-b)^3$.

19. Find the continued product of $-2\,a\,c^2$, $-4\,a\,c^{-3}$, $-3\,a\,b^2$, and $-b\,c\,d$. Ans. $24\,a^3 b^3 d$.

CASE II.

68. When one or both of the factors are polynomials.

1. Required the product of $2x - y$ by $x + y$.

OPERATION.

$$\begin{array}{r} 2x - y \\ x + y \\ \hline 2x^2 - xy \\ \quad 2xy - y^2 \\ \hline 2x^2 + xy - y^2 \end{array}$$

Since the whole expression $2x - y$ is to be multiplied by $x + y$, each of its terms is to be taken x plus y times; x times $2x - y = 2x^2 - xy$; y times $2x - y = 2xy - y^2$,

and the sum of these partial products is $2x^2 + xy - y^2$, the product required. Hence, the

RULE.

Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.

EXAMPLES.

(2.)

$$\begin{array}{r} 3a + 5x \\ 4m \\ \hline 12am + 20mx \end{array}$$

(3.)

$$\begin{array}{r} 8a^2bc - d \\ 5ad^2 \\ \hline 40a^3bcd^2 - 5ad^3 \end{array}$$

(4.)

$$\begin{array}{r} abc + m^n \\ 4am \\ \hline 4a^2bcm + 4am^{n+1} \end{array}$$

(5.)

$$\begin{array}{r} 3a - 2b \\ 2a - 5b \\ \hline 6a^2 - 4ab \\ \quad - 15ab + 10b^2 \\ \hline 6a^2 - 19ab + 10b^2 \end{array}$$

(6.)

$$\begin{array}{r} 1 - x + x^2 - x^3 \\ 1 + x \\ \hline 1 - x + x^2 - x^3 \\ \quad x - x^2 + x^3 - x^4 \\ \hline 1 \qquad\qquad - x^4 \end{array}$$

(7.)

$$\begin{array}{r} 7ax - 4y + 6m \\ 4ay + 2y \\ \hline 28a^2xy - 16ay^2 + 24amy \\ \quad 14axy - 8y^2 + 12my \\ \hline 28a^2xy - 16ay^2 + 14axy - 8y^2 + 24amy + 12my \end{array}$$

8. Multiply $3x^2 - 2xy - y^2$ by $2x - 4y$.
   Ans. $6x^3 - 16x^2y + 6xy^2 + 4y^3$.
9. Multiply $x^2 + 2x + 1$ by $x^2 - 2x + 3$.
   Ans. $x^4 + 4x + 3$.
10. Multiply $a + b - c$ by $a - b + c$.
    Ans. $a^2 - b^2 + 2bc - c^2$.

11. Multiply $3a - 2b$ by $-2a + 4b$.

12. Multiply $a^2 + ab + b^2$ by $a - b$. Ans. $a^3 - b^3$.

13. Multiply $1 - x + x^2 - x^3$ by $a + ax$.

14. Multiply $5a^2 - 3ab + 4b^2$ by $6a - 5b$.

Ans. $30a^3 - 43a^2b + 39ab^2 - 20b^3$.

15. Required the continued product of $3a - x$, $2a + 4x$, and $4a - 2x$.

Ans. $24a^3 + 28a^2x - 36ax^2 + 8x^3$.

69. It is sometimes sufficient to *indicate* the multiplication of polynomials, by inclosing each of the given factors in a parenthesis, and writing them one after the other, with, or without, the sign $\times$ between the parentheses. When the indicated multiplication is performed, the expression is said to be *expanded*, or *developed*.

16. Indicate the multiplication of $2x^2 - 3xy + 6$ by $3x^2 + 3xy - 5$.

Ans. $(2x^2 - 3xy + 6)\ (3x^2 + 3xy - 5)$.

17. Expand $(3a + 4b)\ (2a + b)$.

Ans. $6a^2 + 11ab + 4b^2$.

18. Expand $(a^4 - a^3x + a^2x^2 - ax^3 + x^4)\ (a + x)$.

Ans. $a^5 + x^5$.

19. Develop $(a^4 - x^4) \times (a^4 - x^4)$.

Ans. $a^8 - 2a^4x^4 + x^8$.

20. Develop $(a^m - a^n)\ (2a - a^n)$.

Ans. $2a^{m+1} - 2a^{n+1} - a^{m+n} + a^{2n}$.

21. Show that $(1 + x)\ (1 + x^4)\ (1 - x + x^2 - x^3) = 1 - x^8$.

22. Find the value of the expression $(a + 2x)\ (a - 3x)\ (a + 4x)$. Ans. $a^3 + 3a^2x - 10ax^2 - 24x^3$.

23. Expand $[(a^2 - 3a + 3)\ a - 1] \times [(a - 2)\ a + 1]$.

Ans. $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$.

MULTIPLICATION BY DETACHED COEFFICIENTS.

70. A polynomial is said to be *arranged* according to the *decreasing* powers of any letter, when the term having the highest exponent of the letter is placed first, that having the next highest immediately after, and so on. Thus,

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

is arranged according to the decreasing powers of a.

71. A polynomial is said to be *arranged* according to the *increasing* powers of any letter, when the term having the lowest exponent is placed first, that having the next lowest immediately after, and so on. Thus,

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

is arranged according to the increasing powers of b.

72. When the two factors are each *homogeneous* the product will be homogeneous; and the *degree* of any term will be equal to the sum of the exponents expressing the degrees of its factors.

Hence, if two polynomials contain but two letters, and both are arranged according to the successive powers of the same letter, the product will be arranged according to that letter. Thus,

$$a^3 - a^2b - ab^2 + b^3,$$

is arranged like its two factors, $a^2 - 2ab + b^2$ and $a + b$, according to the decreasing powers of a, and increasing powers of b.

73. When the factors are both arranged according to the powers of the same letter, the *first* and *last* terms of the product are *unlike* any others. These are,

1. The term produced by multiplication of the two terms affected with the *highest* exponent of the same letter.

2. The term produced by multiplication of the two terms affected with the *lowest* exponent of the same letter.

74. The coefficients of a product depend upon the coefficients of the factors, and not upon the literal parts of the terms.

Hence, the coefficients of the factors, *detached* and multiplied together, will give the coefficients of the product; and if to these the proper letters are afterwards annexed, the result will be the entire product.

1. Let it be required to find the product of $3a^2 + 4ab - 5b^2$ by $2a^2 - 6ab + 4b^2$.

OPERATION.

$$
\begin{array}{rl}
3 + 4 - 5 & = \text{Coefficients of Multiplicand,} \\
2 - 6 + 4 & = \text{Coefficients of Multiplier,} \\
\hline
6 + 8 - 10 & \\
- 18 - 24 + 30 & \\
12 + 16 - 20 & \\
\hline
6 - 10 - 22 + 46 - 20 & = \text{Product of Coefficients.}
\end{array}
$$

Since the factors are homogeneous and of the second degree, by Art. 72, the product will be homogeneous and of the fourth degree. Also, since the powers of a decrease, and the powers of b increase, in the factors by unity, the same letters will be arranged in like manner in the product. Therefore, annexing to the coefficients of the product their proper literal parts, we have

$6a^4 - 10a^3b - 22a^2b^2 + 46ab^3 - 20b^4 =$ Entire Product.

2. Let it be required to find the product of $3a^3 - 4ab^2 + 6b^3$ by $2a^2 - 4b^2$.

OPERATION.

$$
\begin{array}{l}
3 + 0 - 4 + 6 \\
2 + 0 - 4 \\
\hline
6 + 0 - 8 + 12 \\
\quad - 12 - 0 + 16 - 24 \\
\hline
6 + 0 - 20 + 12 + 16 - 24
\end{array}
$$

$6a^5 + 0a^4b - 20a^3b^2 + 12a^2b^3 + 16ab^4 - 24b^5 =$
$6a^5 - 20a^3b^2 + 12a^2b^3 + 16ab^4 - 24b^5 =$ Product.

Here, in the first factor, the term containing the second power of a and the first power of b is wanting, and in the other factor that containing the first power of both a and b is wanting; and, for convenience, in keeping dissimilar terms distinct, we put 0 in the place of the coefficient of a wanting term.

But since a, or any quantity, taken 0 times is equal to 0, terms with zero coefficients are valueless and unessential, and may disappear in the final result.

The two preceding examples illustrate the method of *multiplication by detached coefficients.*

EXAMPLES.

3. Multiply $y^2 + y - 3$ by $y^3 - y$.
Ans. $y^5 + y^4 - 4y^3 - y^2 + 3y$.

4. Multiply $x^5 + x^4 + x^3 + x^2 + x + 1$ by $x - 1$.
Ans. $x^6 - 1$.

5. Multiply $a^2 - 2ab + 4b^2$ by $a^2 + 2ab + 4b^2$.
Ans. $a^4 + 4a^2b^2 + 16b^4$.

6. Multiply $3a^4 + 3a^3b + 3a^2b^2 + 3ab^3 + 3b^4$ by $7a - 7b$.
Ans. $21a^5 - 21b^5$.

7. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
Ans. $x^4 - y^4$.

DIVISION.

75. DIVISION, in Algebra, is the process of finding how many times one quantity is contained in another;

Or, it is the process of finding one of two factors, when their product and the other factor are given.

Hence, division is the converse of multiplication.

The *Dividend* is the quantity to be divided.

The *Divisor* is the quantity by which we divide.

The *Quotient* is the result of the division.

76. To determine the law of the COEFFICIENTS of the quotient.

Let it be required to find the quotient of $14ab$ divided by $7a$.

Since the quotient must be a quantity which, multiplied by $7a$, the divisor, will produce $14ab$, the dividend, that quantity is $2b$. Hence,

The COEFFICIENT *of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.*

77. To determine the law of the EXPONENTS of the quotient.

1. Let it be required to find the quotient of a^5 divided by a^3.

Since $a^5 = aaaaa$, and $a^3 = aaa$, it is evident that the quotient, or any quantity which, multiplied by the divisor a^3, will equal the dividend a^5, must be aa, or a^2. Hence,

The EXPONENT *of a letter in the quotient is equal to its exponent in the dividend, diminished by its exponent in the divisor.*

2. Let it be required to divide a^n by a^n.

Then, $\frac{a^n}{a^n} = a^{n-n} = a^0$; but $\frac{a^n}{a^n} = 1$.

Therefore (Ax. 7), $a^0 = 1$; and as a may have any value whatever,

Any quantity whose exponent is ^{0} *is equal to* 1.

By this notation the *trace* of a letter which has disappeared in the operation of division may be preserved. Thus, the quotient of a^2b^3 by a^2b^2, if important to indi-

cate that a originally entered into the terms, may be written

$$a^0 b.$$

3. Let it be required to divide a^2 by a^5.

Then, $\frac{a^2}{a^5} = a^{2-5} = a^{-3}$; that is,

The exponent of a letter in the quotient will be negative, when the exponent of that letter is greater in the divisor than it is in the dividend.

Also, $\frac{a^2}{a^5} = \frac{1}{a^3}$; consequently, $a^{-3} = \frac{1}{a^3}$; that is,

Any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.

So, also, $\frac{a^5}{a^3} = \frac{1}{a^{3-5}} = \frac{1}{a^{-2}}$.

But, $\frac{a^5}{a^3} = a^2$; consequently, $\frac{1}{a^{-2}} = a^2$; hence,

Any factor may be transferred from the divisor to the dividend, or the reverse, by changing the sign of its exponent.

Since the signs + and — indicate opposite relations, processes, or conditions (Art. 34), we should infer, from the signs themselves, that, if a *positive* exponent indicates the number of times a quantity is taken as a *factor*, a *negative* exponent must show the number of times it is used as a *divisor*.

If a negative exponent stands already in a divisor, then the quantity which it affects is a divisor of a divisor, and may be regarded as a factor of the dividend.

The relation of positive and negative exponents to each other, and to the exponent 0, is readily illustrated by such a series as the following, in which the exponents decrease regularly by one, to indicate a division by 3.

3^3,	3^2,	3^1,	3^0,	3^{-1},	3^{-2},	3^{-3}
27,	9,	3,	1,	$\frac{1}{3}$,	$\frac{1}{9}$,	$\frac{1}{27}$

78. To determine the law of the SIGNS of the quotient.

Since the quotient multiplied by the divisor produces the dividend, it follows,

$+$ *divided by* $+$, *and* $-$ *divided by* $-$, *produce* $+$,
$-$ *divided by* $+$, *and* $+$ *divided by* $-$, *produce* $-$.

Thus, $+ab \div +b = +a$, $\quad -ab \div -b = +a$,
$+ab \div -b = -a$, $\quad -ab \div +b = -a$.

Hence, in division, as in multiplication,

Like signs produce $+$, *and unlike signs produce* $-$.

CASE I.

79. When the divisor is a monomial.

From the preceding principles is derived the following

RULE.

Divide the coefficient of the dividend by that of the divisor; and to the result annex the letters of the dividend and divisor, each with an exponent equal to that in the dividend diminished by the exponent in the divisor, omitting all letters whose exponents become zero.

Make the terms in the quotient positive when the dividend and divisor have like signs, and negative when they have different signs.

When the dividend has more than one term, divide each separately, and connect the results by their proper signs.

EXAMPLES.

(1.) $\quad \dfrac{9a^2bcxy}{3abc} = 3axy$

(2.) $\quad \dfrac{12a^3x^2y}{4axz} = 3a^2xyz^{-1}$

(3.)

$$\frac{9a^3b + 6a^4c - 12ab}{3a} = 3a^2b + 2a^3c - 4b$$

4. Divide $6a^4b^2cd$ by $3ab$. Ans. $2a^3bcd$.

5. Divide $12ax^2y^2$ by $4ax^2$.

6. Divide $17a^5xy^7$ by 17. Ans. a^5xy^7.

7. Divide $8a^3bc + 16a^5bc - 4a^2b^2c^2$ by $4a^2c$.
Ans. $2ab + 4a^3b - c$.

8. Divide $9a^5bc - 3a^2b + 18a^3bc$ by $3ab$.

9. Divide $20a^4bc + 15abd^3 - 10a^2b$ by $5ab$.
Ans. $4a^3c + 3d^3 - 2a$.

10. Divide $-6bx^3y^2z$ by $2bx^3y$. Ans. $-3yz$.

11. Divide a^{m-n} by a^{-m}. Ans. a^{2m-n}.

12. Divide $6(a+b)^3x^2y$ by $3(a+b)^2xy$.
Ans. $2(a+b)x$.

13. Divide $3a^3(a-b) + 9a(a+b)$ by $3a$.
Ans. $a^2(a-b) + 3(a+b)$.

14. Divide $x^2y^2 - xy$ by xy. Ans. $xy - 1$.

15. Divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$. Ans. $a^{\frac{1}{6}}$.

16. Divide $15(x+y)^2 - 5a(x+y) + 10b(x+y)$ by $-5(x+y)$. Ans. $-3(x+y) + a - 2b$.

17. Divide $12a^{-2} - 8a^2b + 16a^3x - 10a^{-2}y$ by $2a^2$.
Ans. $6a^{-4} - 4b + 8ax - 5a^{-4}y$.

18. Divide $ax^n + ax^{-n} + ax^{n+2} + ax^{n+m}$ by x^n.
Ans. $a + ax^{-2n} + ax^2 + ax^m$.

CASE II.

80. When the divisor is a polynomial.

1. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.

OPERATION.

$$
\begin{array}{l|ll}
a^3 + 5a^2x + 5ax^2 + x^3 & a + x, & \text{Divisor.} \\
\underline{a^3 + a^2x} & \overline{a^2 + 4ax + x^2}, & \text{Quotient.} \\
4a^2x + 5ax^2 & & \\
\underline{4a^2x + 4ax^2} & & \\
ax^2 + x^3 & & \\
\underline{ax^2 + x^3} & &
\end{array}
$$

The divisor and dividend are both arranged with reference to the decreasing powers of a.

Now, since the first term of the dividend, as arranged, must equal the product of the first term of the divisor by the first term of the quotient (Art. 73), we have a^2 for the first term of the quotient.

The product of the divisor by this term, subtracted from the dividend, gives a *new* dividend.

With this dividend, *arranged as the original one*, we proceed in like manner as with that, and obtain $4ax$, the next term of the quotient; and so on.

The divisor is written on the right of the dividend, for the convenience of multiplying it by the several terms of the quotient, as they are found. It may be written on the left.

Hence, the

RULE.

Arrange both dividend and divisor according to the decreasing powers of some letter.

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.

Multiply the whole divisor by this term, and subtract the product from the dividend.

Regard the remainder as a new dividend, arrange it and

find the next term of the quotient, in the same manner as before: and so proceed until all the terms of the dividend are exhausted.

When it happens that the division *cannot be exactly performed*, the quantity remaining undivided may be written with the divisor under it, in the form of a fraction, and annexed to the quotient.

In general, division *terminates* when the first term of the arranged remainder will not contain the first term of the divisor.

EXAMPLES.

(2.)

$$\begin{array}{l|l} a^4 + 4a^2b^2 + 16b^4 & a^2 - 2ab + 4b^2 \\ a^4 - 2a^3b + 4a^2b^2 & \overline{a^2 + 2ab + 4b^2} \\ \hline 2a^3b + 16b^4 & \\ 2a^3b - 4a^2b^2 + 8ab^3 & \\ \hline 4a^2b^2 - 8ab^3 + 16b^4 & \\ 4a^2b^2 - 8ab^3 + 16b^4 & \\ \hline \end{array}$$

(3.)

$$\begin{array}{l} 4x^2-7x\,)\,12x^5-13x^4-14x^3+8x^2+1\,(\,3x^3+2x^2+2+\frac{14x+1}{4x^2-7x} \\ \quad 12x^5-21x^4 \\ \quad \overline{\qquad 8x^4-14x^3} \\ \qquad\quad 8x^4-14x^3 \\ \qquad\quad \overline{\qquad\quad 8x^2+1} \\ \qquad\qquad 8x^2-14x \\ \qquad\qquad \overline{\qquad 14x+1}, \text{ Remainder.} \end{array}$$

Here, the divisor is written on the left of the dividend.

4. Divide $8a^3 - 4a^2b - 6ab^2 + 3b^3$ by $2a - b$.

Ans. $4a^2 - 3b^2$.

5. Divide $3b^3 + 3ab^2 - 4a^2b - 4a^3$ by $a + b$.

Ans. $-4a^2 + 3b^2$.

6. Let $2a^2x^2 - 5ax + 2$ be divided by $2ax - 1$.

Ans. $ax - 2$.

7. Divide $21a^5 - 21b^5$ by $7a - 7b$.

Ans. $3a^4 + 3a^3b + 3a^2b^2 + 3ab^3 + 3b^4$.

8. Divide $x^4 - y^4 + 2y^2z^2 - z^4$ by $x^2 + y^2 - z^2$.

Ans. $x^2 - y^2 + z^2$.

9. Divide $x^4 + y^4$ by $x + y$.

Ans. $x^3 - x^2y + xy^2 - y^3 + \frac{2y^4}{x+y}$.

10. Divide $a^3 + 5a + 5a^{-1} + a^{-3}$ by $a + a^{-1}$.

Ans. $a^2 + 4 + a^{-2}$.

11. Divide $a^3 - x^3$ by $a + x$.

Ans. $a^2 - ax + x^2 - \frac{2x^3}{a+x}$.

12. Divide $x^{n+1} + x^ny + xy^n + y^{n+1}$ by $x^n + y^n$.

Ans. $x + y$.

13. Divide $x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$.

Ans. $x^2 - 3x - y$.

14. Divide $1 + a$ by $1 - a$.

Ans. $1 + 2a + 2a^2 + 2a^3 +$, etc.

In examples of this kind the division *does not terminate*, there being a remainder, however far the operation may be carried.

15. Divide a by $1 + x$.

Ans. $a - ax + ax^2 - ax^3 +$, etc.

16. Divide $a^{m+n} + a^{n+1}b^{m-1} - a^{m-1}b^{n+1} - b^{m+n}$ by $a^{n+1} - b^{n+1}$. Ans. $a^{m-1} + b^{m-1}$.

17. Divide $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$ by $a^4 + a^3b + a^2b^2 + ab^3 + b^4$. Ans. $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

DIVISION BY DETACHED COEFFICIENTS.

81. Division, as well as multiplication (Art. 74), sometimes may be conveniently performed by operating on the

coefficients *detached*, and afterwards supplying the proper literal parts. Thus.

1. Let it be required to divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.

OPERATION.

$1 + 3 + 3 + 1 \mid 1 + 1$ = Coefficients of the Divisor.
$1 + 1 \quad\quad\quad\quad 1 + 2 + 1$ = Coefficients of the Quotient.
$2 + 3 \quad a^2 + 2ab + b^2$ = Quotient.
$2 + 2$
$1 + 1$
$1 + 1$

The coefficients of the quotient are 1, 2, and 1; then, since $a^3 \div a = a^2$, and $b^3 \div b = b^2$, we have for the quotient $1a^2 + 2ab + 1b^2$, or $a^2 + 2ab + b^2$.

2. Let it be required to divide $x^4 - y^4$ by $x^2 - y^2$.

OPERATION.

$1 + 0 + 0 + 0 - 1 \mid 1 + 0 - 1$
$1 + 0 - 1 \quad\quad 1 + 0 + 1$
$1 + 0 - 1 \quad x^2 + 0xy + y^2 = x^2 + y^2$ = Quotient.
$1 + 0 - 1$

Here, the place of wanting powers is supplied by zero; and in the second term of the quotient the coefficient being zero, that term is suppressed (Art. 74).

EXAMPLES.

3. Divide $6x^3 - 17x^2y + 16y^3$ by $3x - 4y$.

Ans. $2x^2 - 3xy - 4y^2$.

4. Divide $3y^3 + 3xy^2 - 4x^2y - 4x^3$ by $x + y$.

Ans. $3y^2 - 4x^2$.

5. Divide $a^4 - 3a^3b - 8a^2b^2 + 18ab^3 - 8b^4$ by $a^2 + 2ab - 2b^2$. Ans. $a^2 - 5ab + 4b^2$.

6. Divide $m^5 - 5m^4n + 10m^3n^2 - 10m^2n^3 + 5mn^4 - n^5$ by $m^2 - 2mn + n^2$. Ans. $m^3 - 3m^2n + 3mn^2 - n^3$.

7. Divide $x^7 + y^7$ by $x + y$.

Ans. $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$.

GENERAL PRINCIPLES.

82. Algebra may be divided into two parts, *Arithmetical* and *Symbolical*.

83. In ARITHMETICAL ALGEBRA symbols are restricted in their use to denoting the numbers and operations which occur in Arithmetic.

84. In SYMBOLICAL ALGEBRA the rules of Arithmetical Algebra are assumed to hold universally, and it is then determined what must be denoted by the symbols and the operations, in order to insure this extension.

85. The ABSOLUTE VALUE of a quantity is the number represented by the quantity, taken independently of the sign affecting it. Thus, 1 and -1 have the same absolute value.

In *algebraic language*, however, 1 is greater than -1, and -2 than -3.

86. In general, of two negative quantities, algebraically considered, that is the *greater* which has the less number of units, or which has the smaller absolute value.

Hence the signification of *sum* (Art. 52) and of *difference* (Art. 58), and the like extension of the meaning of all the fundamental operations.

87. In reducing similar terms to an equivalent term, the operation affects the coefficient only, and never the exponent. For a coefficient denotes the number of times

a quantity is taken *additively*, and an exponent the number of times it is taken as a *factor*. Thus

$$3\,a^2b^3 - 7\,a^2b^3 - 6\,a^2b^3 = -10\,a^2b^3.$$

88. The *whole number of terms* in the product of two polynomials, when there is no reduction of similar terms, is equal to the product of the number of terms in the one factor by the number of terms in the other. (Art. 68.)

Thus, in general, if there are m terms in one factor and n terms in another, the whole number of terms in their product will be

$$m \times n, \text{ or } mn.$$

By reduction of similar terms the whole number of terms in a product may be made often much less, but never less than two. (Art. 73.)

USEFUL FORMULAS.

89. A FORMULA is an algebraic expression of a general rule.

90. The following formulas will be found very useful in abridging algebraic operations.

If a and b represent any two quantities whatever, then,

$$(a+b)^2 = (a+b)\ (a+b) = a^2 + 2ab + b^2. \quad (1)$$

That is,

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

Again,

$$(a-b)^2 = (a-b)\ (a-b) = a^2 - 2ab + b^2. \quad (2)$$

That is,

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.

Also,

$$(a + b)\ (a - b) = a^2 - b^2. \qquad (3)$$

That is,

The product of the sum and difference of two quantities is equal to the difference of their squares.

EXAMPLES.

1. Required the square of $3a + 2b$.

By (1),

$$(3a + 2b)^2 = 9a^2 + 12ab + 4b^2.$$

2. Required the square of $3a - 2b^2$.

By (2),

$$(3a - 2b^2)^2 = 9a^2 - 12ab^2 + 4b^4.$$

3. Required the product of $6a + b$ by $6a - b$.

By (3),

$$(6a + b)\ (6a - b) = 36a^2 - b^2$$

4. Expand $(x + 2y)\ (x + 2y)$.

Ans. $x^2 + 4xy + 4y^2$.

5. Expand $(2m + 3n)\ (2m + 3n)$.

6. Expand $(3a^2 - b^3)\ (3a^2 - b^3)$.

Ans. $9a^4 - 6a^2b^3 + b^6$.

7. Expand $(4ab - x)\ (4ab - x)$.

8. Required the product of $1 - 2x^2$ by itself.

Ans. $1 - 4x^2 + 4x^4$.

9. Multiply $9abc - 10ay^2$ by $9abc + 10ay^2$.

Ans. $81a^2b^2c^2 - 100a^2y^4$.

10. Expand $(3a^2 - b^3)\ (3a^2 + b^3)$.

11. Expand $(x^4 - y^2)\ (x^4 - y^2)$.

Ans. $x^8 - 2x^4y^2 + y^4$.

12. Expand $(3 - a^n)\ (3 - a^n)$. Ans. $9 - 6a^n + a^{2n}$.

13. Expand $(ab^{-1} - a^{-1}b)^2$.

Ans. $a^2b^{-2} - 2 + a^{-2}b^2$.

14. Multiply $1 + (a - b)$ by $1 - (a - b)$.

Ans. $1 - (a - b)^2$, or $1 - a^2 + 2ab - b^2$.

15. Expand $(a + b + c)(a + b - c)$.

$$(a + b + c)(a + b - c) = (\overline{a + b} + c)(\overline{a + b} - c)$$
$$= (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2.$$

16. Expand $(c - a + b)(c + a - b)$.

Ans. $c^2 - a^2 + 2ab - b^2$.

17. Multiply $(a + b) + (c + d)$ by $(a + b) - (c + d)$.

Ans. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$.

FACTORING.

91. FACTORING is the process of resolving a quantity into its factors.

92. The FACTORS of a quantity are such integral quantities as will divide the quantity without a remainder.

93. A PRIME QUANTITY is one that cannot be divided, without a remainder, by any integral quantity different from itself, or unity.

Thus, a, b, and $a + c$ are prime quantities.

94. Quantities are said to be *prime to each other*, when they have no common factor greater than unity.

95. One quantity is said to be *divisible* by another when the latter will divide the former without a remainder.

Thus, ab, and $ab + a^2b^2$, are each divisible by a, b, and ab.

PRINCIPLES OF FACTORING.

96. *The difference of any two equal powers of two quantities is always divisible by the difference of the quantities.*

For, let a and b represent any two quantities what-

ever, whose common exponent is m. Then, $a^m - b^m$ is divisible by $a - b$.

Commencing the division, we have

$$\begin{array}{l|l} a^m - b^m & a - b \\ \hline a^m - a^{m-1}b & \overline{a^{m-1} + a^{m-2}b \ldots} \\ \hline a^{m-1}b - b^m = b\,(a^{m-1} - b^{m-1}) & = \text{First Remainder.} \\ a^{m-1}b - a^{m-2}b^2 & \\ \hline a^{m-2}b^2 - b^m = b^2\,(a^{m-2} - b^{m-2}) & = \text{Second Remainder.} \end{array}$$

Dividing a^m by a, we have for the first term of the quotient a^{m-1}, and for the first remainder $a^{m-1}b - b^m$, which may be put under the form of

$$b\ (a^{m-1} - b^{m-1}).$$

Now, it is evident, if this remainder, whose factors are b and $a^{m-1} - b^{m-1}$, be divisible by $a - b$, the dividend $a^m - b^m$ will be exactly divisible by $a - b$; that is,

If the difference of the same powers of two quantities is divisible by the difference of those quantities, then the difference of the next higher powers of the same quantities is also divisible by that difference.

But $a^2 - b^2$ is divisible by $a - b$, or

$$(a^2 - b^2) \div (a - b) = a + b\ ;$$

hence $a^3 - b^3$ is divisible by $a - b$; and since $a^3 - b^3$ is divisible by $a - b$, it follows that $a^4 - b^4$ is divisible by the same, and so on to any power m.

Continuing the division of $a^m - b^m$ by $a - b$, we have

$$\left.\begin{array}{l} (a^m - b^m) \div (a - b) = a^{m-1} + a^{m-2}b + a^{m-3}b^2 \ldots \\ \qquad\qquad + a^2 b^{m-3} + a\,b^{m-2} + b^{m-1}\ ; \end{array}\right\} \quad (1)$$

which may be verified by multiplication.

In (1) give particular values to m, and we have

$$\left.\begin{aligned}(a^2 - b^2) \div (a - b) &= a + b,\\ (a^3 - b^3) \div (a - b) &= a^2 + ab + b^2,\\ (a^4 - b^4) \div (a - b) &= a^3 + a^2 b + ab^2 + b^3,\\ (a^5 - b^5) \div (a - b) &= a^4 + a^3 b + a^2 b^2 + ab^3 + b^4.\end{aligned}\right\} (2)$$

In (2) let b or a become 1; then,

$$\left.\begin{aligned}(a^2 - 1) \div (a - 1) &= a + 1,\\ (a^3 - 1) \div (a - 1) &= a^2 + a + 1,\\ (a^4 - 1) \div (a - 1) &= a^3 + a^2 + a + 1; \text{ and}\\ (1 - b^2) \div (1 - b) &= 1 + b,\\ (1 - b^3) \div (1 - b) &= 1 + b + b^2,\\ (1 - b^4) \div (1 - b) &= 1 + b + b^2 + b^3,\end{aligned}\right\} (3)$$

and so on.

97. It may also be proved that *the difference of any two equal even powers of two quantities is always divisible by the sum of the quantities.*

Thus,

$$\left.\begin{aligned}(a^2 - b^2) \div (a + b) &= a - b,\\ (a^4 - b^4) \div (a + b) &= a^3 - a^2 b + ab^2 - b^3,\\ (a^6 - b^6) \div (a + b) &= a^5 - a^4 b + a^3 b^2 - a^2 b^3 + ab^4 - b^5.\end{aligned}\right\} (1)$$

Also,

$$\left.\begin{aligned}(a^2 - 1) \div (a + 1) &= a - 1,\\ (a^4 - 1) \div (a + 1) &= a^3 - a^2 + a - 1,\\ (a^6 - 1) \div (a + 1) &= a^5 - a^4 + a^3 - a^2 + a - 1; \text{ and}\\ (1 - b^2) \div (1 + b) &= 1 - b,\\ (1 - b^4) \div (1 + b) &= 1 - b + b^2 - b^3,\\ (1 - b^6) \div (1 + b) &= 1 - b + b^2 - b^3 + b^4 - b^5,\end{aligned}\right\} (2)$$

and so on.

98. It may likewise be proved that *the sum of two equal odd powers of two quantities is always divisible by the sum of the quantities.*

Thus,

$$\left.\begin{aligned}(a^3 + b^3) \div (a + b) &= a^2 - ab + b^2,\\ (a^5 + b^5) \div (a + b) &= a^4 - a^3 b + a^2 b^2 - ab^3 + b^4.\end{aligned}\right\} (1)$$

$$\left.\begin{array}{l} \text{Also, } (a^3+1) \div (a+1) = a^2 - a + 1, \text{ and} \\ \qquad (1+b^3) \div (1+b) = 1 - b + b^2, \end{array}\right\} \quad (2)$$

and so on.

99. The foregoing principles may be enunciated thus : —

When the two equal powers are ODD, *their* DIFFERENCE *is divisible by the* DIFFERENCE *of the quantities, and their* SUM *by the* SUM *of the quantities.*

When the two equal powers are EVEN, *their* DIFFERENCE *is divisible by* BOTH *the difference and the sum of the quantities, but their* SUM *is divisible by* NEITHER *the difference nor the sum of the quantities.*

NOTE. The quantity $a^{2n} + b^{2n}$ may always be factored when n contains an odd factor. Thus, $a^6 + b^6$ is divisible by $a^2 + b^2$, and $a^{12} + b^{12}$ by $a^4 + b^4$; but $a^2 + b^2$ and $a^4 + b^4$ are prime.

100. The factoring of monomials may be performed by inspection, since their prime factors are evidently those of their coefficients, and each letter of the expression taken as many times as there are units in its exponent. Thus,

$$12\, a^2 b = 2 \times 2 \times 3\, a\, a\, b.$$

But the decomposition of polynomials often requires a resort to trial, or the law of their formation, and the exercise of much skill.

101. The following observations will aid in factoring some polynomials ; but with regard to others, no specific directions can be given : —

1. When the terms of a polynomial have a common monomial factor, it may be written as one of the factors, with the quotient obtained by dividing the given polynomial by this factor as another. Thus,

$$6x^4 + 3x^3 - 9x = 3x\,(2x^3 + x^2 - 3).$$

2. Any trinomial can be resolved into two equal binomial factors, by Formulas 1 and 2, Art. 90, when two of the terms are squares and positive, and the other term is twice the product of their square roots. Thus,

$$a^2 + 2ab + b^2 = (a + b)(a + b);$$
$$a^2 - 4ab^2 + 4b^4 = (a - 2b^2)(a - 2b^2).$$

3. Any binomial can be resolved into two binomial factors, by Formula 3, Art. 90, when its terms represent the difference between two squares. Thus,

$$4x^2 - y^2 = (2x + y)(2x - y).$$

4. Any binomial which consists of the *difference* of any two equal powers, or the *sum* of any two equal *odd* powers, may be factored by Articles 96, 97, and 98; for the quotient and divisor are factors of the dividend. Thus,

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2);$$
$$a^3 + 1 = (a + 1)(a^2 - a + 1);$$
$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

EXAMPLES.

1. Factor $12am + 20mx$.
   Ans. $4m(3a + 5x)$, or $2 \times 2m(3a + 5x)$.
2. Factor $a^2 - 2a + 1$. Ans. $(a - 1)(a - 1)$.
3. Required the binomial factors of $a^2 + 2ax + x^2$.
4. Factor $m^3nx + m^2n^2y + m^2nxy$.
   Ans. $m^2n(mx + ny + xy)$.
5. Resolve $3ab^2 + 9a^2bd$ into its factors.
6. Resolve $1 - 8x^3$ into its factors.
   Ans. $(1 - 2x)(1 + 2x + 4x^2)$.
7. Factor $3bx^2 + 6bxy + 3by^2$.
   Ans. $3b(x + y)(x + y)$.

8. Factor $81\,a^8 - 1$.

Ans. $(9\,a^4 + 1)\,(3\,a^2 + 1)\,(3\,a^2 - 1)$.

9. Factor $5\,x^5 + 5\,y^5$.

Ans. $5\,(x + y)\,(x^4 - x^3 y + x^2 y^2 - x y^3 + y^4)$.

10. Factor $n\,x^2 - 2\,n\,x\,y + n\,y^2$.

Ans. $n\,(x - y)\,(x - y)$.

11. Factor $a^{-2} + 2\,a^{-1}\,b^{-1} + b^{-2}$.

Ans. $(a^{-1} + b^{-1})\,(a^{-1} + b^{-1})$.

12. Factor $x^8 - y^8$. Ans. $(x^4 + y^4)\,(x^2 + y)\,(x^2 - y)$.

13. Factor $a^2 + 2\,a\,b + b^2 - c^2$.

$$a^2 + 2\,a\,b + b^2 - c^2 = (a + b)^2 - c^2$$
$$= (\overline{a + b} + c)\,(\overline{a + b} - c) = (a + b + c)\,(a + b - c).$$

14. Factor $c^2 - (a^2 - 2\,a\,b + b^2)$.

Ans. $(c - a + b)\,(c + a - b)$.

15. Resolve $a\,d + a\,y + d\,x + x\,y$ into its factors.

$$a\,d + a\,y + d\,x + x\,y = a\,(d + y) + x\,(d + y)$$
$$= (a + x)\,(d + y).$$

16. Factor $a\,m - b\,n + b\,m - a\,n$.

Ans. $(a + b)\,(m - n)$.

17. Resolve $a^6 - b^6$ into its factors.

Ans. $(a + b)\,(a^2 - a\,b + b^2)\,(a - b)\,(a^2 + a\,b + b^2)$.

102. A trinomial may be resolved into two *binomial* factors, when *one of the terms is a square and another the product of the algebraic sum of the two factors of a third term by the square root of the first.*

For, if we multiply $(x + a)$ by $(x + b)$, we have

$$x^2 + (a + b)\,x + a\,b\,; \qquad (1)$$

Or, if we multiply $(x - a)$ by $(x - b)$, we have

$$x^2 - (a + b)\,x + a\,b. \qquad (2)$$

Also, if we multiply $(x + a)$ by $(x - b)$, or $(x - a)$ by $(x + b)$, we have

$$x^2 + (a - b)\, x - a\, b, \qquad (3)$$

$$x^2 + (b - a)\, x - a\, b. \qquad (4)$$

18. Factor $x^2 + 12\, x + 20$. Ans. $(x + 10)\ (x + 2)$.

19. Factor $a^2 - 16\, a + 64$. Ans. $(a - 8)\ (a - 8)$.

20. Factor $a^2 + a - 2$. Ans. $(a - 1)\ (a + 2)$.

21. Factor $b^4 - b^2 - 20$. Ans. $(b^2 + 4)\ (b^2 - 5)$.

22. Factor $x^2 y^2 + 12\, x\, y + 27$.

Ans. $(x\, y + 9)\ (x\, y + 3)$.

23. Resolve $x^2 + 3\frac{1}{2}\, x + \frac{3}{2}$ into two binomial factors.

Here $\frac{3}{2} = 3 \times \frac{1}{2}$, and $3\frac{1}{2} = 3 + \frac{1}{2}$;

$\therefore\ x^2 + 3\frac{1}{2}\, x + \frac{3}{2} = (x + 3)\ (x + \frac{1}{2})$.

24. Factor $x^2 - 1\frac{3}{4}\, x + \frac{3}{4}$. Ans. $(x - 3)\ (x - \frac{1}{4})$.

25. Factor $x^2 y - 1\frac{3}{3}\, x\, y - 1\frac{0}{3}\, y$.

Ans. $y\,(x - 5)\ (x + \frac{2}{3})$.

26. Resolve $5\, a^3 b - 5\, a^2 b - 30\, a\, b$ into its factors.

Ans. $5\, a\, b\ (a - 3)\ (a + 2)$.

GREATEST COMMON DIVISOR.

103. A Common Divisor or Measure of two or more quantities is a quantity that will divide each of them without a remainder.

Hence, *any factor common to two or more quantities is a common divisor of those quantities.*

Also, when quantities are prime to each other, they have no common measure greater than unity.

104. The Greatest Common Divisor of two or more quantities is the greatest quantity that will divide each of them without a remainder.

Hence, *the greatest common divisor of two or more quantities is the product of all the prime factors common to those quantities.*

By the *greatest* of two or more algebraic quantities, it may be remarked, is meant the *highest*, with reference to the *coefficients* and *exponents* of the same letters.

105. *The greatest common divisor of two or more quantities is the same as the greatest common divisor of the least of those quantities, and the remainder, or remainders, if any, after the division of the least into the other, or each of the others.*

For, let a and b be two quantities, of which b is the least, and m some integral number.

Suppose, now, that b is not contained in a an exact number of times, but m times, with a remainder, r. Then, since the dividend is equal to the product of the divisor by the quotient, plus the remainder, we have

$$a = mb + r.$$

Also, since the remainder is equal to the dividend minus the product of the divisor by the quotient,

$$r = a - mb.$$

Now, any quantity that will exactly divide b will exactly divide m times b, or mb; and any quantity that will exactly divide b and r will exactly divide mb and r, and consequently will exactly divide their sum, $mb + r$, or its equal, a. Hence, any quantity that is a common divisor of b and r is also a common divisor of a and b.

Again, any quantity that will exactly divide a and b will exactly divide a and mb, and consequently will exactly divide their difference, $a - mb$, or its equal, r. Therefore, any common divisor of a and b must also be a common divisor of b and r.

But the converse of this has already been proved; consequently, the common divisors of a and b, and of b and r, must be identical, and the *greatest* common divisor of a and b must be also the *greatest* common divisor of b and r; which was to be proved.

In like manner, the same may be proved for more than two quantities.

106. In determining the greatest common divisor of algebraic quantities, it is convenient to distinguish two cases.

CASE I.

107. When the quantities can be factored by inspection.

1. Find the greatest common divisor of $3a^2 - 3b^2$ and $3a^2 + 6ab + 3b^2$.

OPERATION.

$$3a^2 - 3b^2 = 3(a + b)(a - b)$$
$$3a^2 + 6ab + 3b^2 = 3(a + b)(a + b)$$
$$3(a + b)$$

Factoring the quantities, we find that 3 and $(a + b)$ are their only common prime factors; then, the product of these (Art. 104), or $3(a + b)$, is the required greatest common divisor. Hence the

RULE.

Resolve the quantities into their prime factors, and find the product of all the factors common to the several quantities.

Any factor forming a part of the greatest common divisor will take the lowest exponent with which it occurs in either of the given quantities.

EXAMPLES.

2. Find the greatest common divisor of $3ax^2 - 2a^2x$ and $a^2x^2 - 3abx$. Ans. ax.

3. Find the greatest common divisor of $4a^2bc$, $6a^2bd$, and $2a^4b^3c$. Ans. $2a^2b$.

4. Find the greatest common divisor of $6x^3y - 12x^2y^2 + 3xy^3$ and $4ax^2 + 4axy + 4a^2x$.

5. Required the greatest common measure of $x^3 - a^3$ and $x^2 - a^2$. Ans. $x - a$.

6. Required the greatest common factor of $m^2 + 2mn + n^2$ and $m^2 - n^2$. Ans. $m + n$.

7. Required the greatest common divisor of $5a^3 - 10a^2b + 5ab^2$ and $8a^3 - 8a^2b$. Ans. $a(a - b)$.

8. Required the greatest common divisor of $b^2c + 7bc + 12c$ and $3ab^2 + 21ab + 36a$.

Ans. $(b + 3)(b + 4)$.

9. Required the greatest common divisor of $x^4 - 1$, $x^5 + x^3$, and $x^4 + 2x^2 + 1$. Ans. $x^2 + 1$.

CASE II.

108. When the quantities cannot readily be factored by inspection.

1. Find the greatest common divisor of $x^2 + 2x + 1$ and $x^3 + 2x^2 + 2x + 1$.

OPERATION.

$$
\begin{array}{l}
x^2 + 2x + 1)\ x^3 + 2x^2 + 2x + 1\ (x \\
\qquad\quad x^3 + 2x^2 + \ x \\
\qquad\quad \underline{\qquad\qquad\qquad} \\
\qquad\qquad\quad x + 1)\ x^2 + 2x + 1\ (x + 1 \\
\qquad\qquad\qquad\ \ x^2 + \ x \\
\qquad\qquad\qquad\ \ \underline{\qquad\qquad} \\
\qquad\qquad\qquad\qquad\ \ x + 1 \\
\qquad\qquad\qquad\qquad\ \ x + 1 \\
\qquad\qquad\qquad\qquad\ \ \underline{\qquad\qquad}
\end{array}
$$

It is evident that if $x^2 + 2x + 1$ will exactly divide $x^3 + 2x^2 + 2x + 1$, it will be the greatest common divisor, since no quantity can have a divisor greater than itself. But we find that the latter is not divisible by the former, there being a remainder, $x + 1$. Now, we know

that the greatest common divisor cannot be greater than this remainder; for the greatest common divisor of two quantities must be a divisor of their remainder after division (Art. 105). We therefore divide the divisor by the remainder, which it exactly divides. As $x+1$ is the greatest common divisor of the remainder and divisor, it must also be that of the divisor and dividend (Art. 105); consequently it is the greatest common divisor required.

2. Find the greatest common divisor of $36a^6 - 18a^5 - 27a^4 + 9a^3$ and $27a^5b^2 - 18a^4b^2 - 9a^3b^2$.

OPERATION.

$$36a^6 - 18a^5 - 27a^4 + 9a^3 = 9a^3(4a^3 - 2a^2 - 3a + 1)$$
$$27a^5b^2 - 18a^4b^2 - 9a^3b^2 = 9a^3b^2(3a^2 - 2a - 1)$$

$$\begin{array}{r|l} & 4a^3 - 2a^2 - 3a + 1 \\ & \quad 3 \\ \hline 3a^2 - 2a - 1) & 12a^3 - 6a^2 - 9a + 3\ (4a \\ & 12a^3 - 8a^2 - 4a \\ \hline & \quad 2a^2 - 5a + 3 \\ & \quad 3 \\ \hline & 6a^2 - 15a + 9\ (2 \\ & 6a^2 - 4a - 2 \\ \hline & \quad -11a + 11 \end{array}$$

$$\begin{array}{r|l} & a - 1) \ 3a^2 - 2a - 1\ (3a + 1 \\ & \quad 3a^2 - 3a \\ \hline & \quad a - 1 \\ & \quad a - 1 \\ \hline \end{array}$$

$9a^3(a-1)$ = the greatest common divisor.

Here, $9a^3$, being common to both the given quantities, is, therefore, a factor of the greatest common divisor (Art. 103), and may be reserved to become a factor of the required greatest common divisor.

The factor b^2, being common only to the terms of the second quantity, may be canceled, since it can form no part of the common divisor.

In proceeding with the other factors, to avoid a fractional quotient, the first is multiplied by 3, which, not being a factor of the second, cannot affect their greatest common divisor. The same is true of the first remainder.

The factor -11 of the remainder $-11a+11$ is not found in $3a^2-2a-1$, and can therefore form no part of the common divisor; hence, we cancel it, and as $a-1$ divides $3a^2-2a-1$ without a remainder, $9a^3(a-1)$ is the required greatest common divisor.

From the preceding solutions is deduced the following

RULE.

Divide the greater quantity by the less, and if there is no remainder, the less quantity will be the divisor required.

If there is a remainder, divide the divisor by it, and continue thus to make the preceding divisor the dividend and the remainder the divisor, until a divisor is obtained which leaves no remainder; the last divisor will be the greatest common divisor.

When the quantities have a common monomial factor, it may be reserved as a factor of the common divisor required.

If in the operation a monomial factor is found common to all the terms of one of the quantities and not of the other, it may be canceled; and, when necessary, the dividend may be multiplied by a factor not common to all the terms of the divisor.

When more than two quantities are given, find the greatest common divisor of two of them, and then of that common divisor and the third of the quantities, and so on. The last divisor will be the greatest common divisor required.

EXAMPLES.

3. Find the greatest common divisor of $a^3 - a^2x + 3ax^2 - 3x^3$ and $a^2 - 5ax + 4x^2$. Ans. $a - x$.

4. Find the greatest common divisor of $x^2 - 7x + 10$ and $4x^3 - 25x^2 + 20x + 25$. Ans. $x - 5$.

5. What is the greatest common factor of $y^4 - x^4$ and $y^3 - y^2x - yx^2 + x^3$? Ans. $y^2 - x^2$.

6. Find the greatest common measure of $x^4 - x^3 + 2x^2 + x + 3$ and $x^4 + 2x^3 - x - 2$.

Ans. $x^2 + x + 1$.

7. Find the greatest common measure of $2a^4 + 3a^3x - 9a^2x^2$ and $6a^4x - 17a^3x^2 + 14a^2x^3 - 3ax^4$.

8. Required the greatest common factor of $x^4 + 6x^3 + 11x^2 + 4x - 4$ and $x^4 + 2x^3 - 5x^2 - 12x - 4$.

Ans. $x^2 + 4x + 4$.

9. Find the greatest common divisor of $6x^2y + 4xy^2 - 2y^3$ and $8x^3 + 4x^2y - 4xy^2$. Ans. $2(x + y)$.

10. Find the greatest common divisor of $6x^2 - x - 5$, $21x^3 - 26x^2 + 5x$, and $4x^3 - 12x^2 + 9x - 1$.

Ans. $x - 1$.

LEAST COMMON MULTIPLE.

109. A MULTIPLE of a quantity is any quantity that can be divided by it without a remainder.

Hence, *a multiple of a quantity must contain all the prime factors of that quantity.*

110. A COMMON MULTIPLE of two or more quantities is one that can be divided by each of them without a remainder.

Hence, *a common multiple of two or more quantities must contain all the prime factors of each of the quantities.*

111. The LEAST COMMON MULTIPLE of two or more quantities is the least quantity that can be divided by each of them without a remainder.

Hence, *the least common multiple of two or more quantities must be the product of all their different prime factors, each taken only the greatest number of times it is found in any one of those quantities.*

112. *If the product of two quantities be divided by their greatest common divisor, the quotient will be their least common multiple.*

For, since the greatest common divisor of two quantities is composed of all the factors common to those quantities (Art. 104), these factors will enter twice into the product of the quantities. Hence, if the product be divided by the greatest common divisor, the quotient will contain only the factors *common* to the quantities, and those *peculiar to each* of them. Now these are the factors of the least common multiple. (Art. 111.)

CASE I.

113. When the quantities can be factored by inspection.

1. Find the least common multiple of $x^2 - a^2$ and $x^3 - a^3$.

OPERATION.

$$\begin{aligned} x^2 - a^2 &= (x - a)\ (x + a) \\ x^3 - a^3 &= (x - a)\ (x^2 + ax + a^2) \\ \hline & (x + a)\ (x - a)\ (x^2 + ax + a^2) \end{aligned}$$

Factoring the quantities, and finding the product of all their different prime factors, we have (Art. 111) for their

least common multiple $(x + a)\ (x - a)\ (x^2 + a\,x + a^2)$. Hence the

RULE.

Resolve the quantities into their prime factors; and the product of these, taking each factor only the greatest number of times it enters into any one of the quantities, will be the least common multiple.

When quantities are prime to each other, their product is their least common multiple.

EXAMPLES.

2. Find the least common multiple of $8\,a^4$, $10\,a^3\,b$, and $12\,a^2\,b^2$. Ans. $120\,a^4\,b^2$.

3. Find the least common multiple of $2\,(a + b)$ and $3\,(a^2 - b^2)$. Ans. $6\,(a^2 - b^2)$.

4. Find the least common multiple of $(x + 1)\ (x^2 - 1)$ and $x^3 - 1$. Ans. $(x + 1)^2\ (x^3 - 1)$.

5. Find the least common multiple of $(a + b)^2$, $a^2 - b^2$, $(a - b)^2$, and $a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3$.

Ans. $(a + b)^3\ (a - b)^2$, or $(a + b)\ (a^2 - b^2)^2$.

6. Find the least common multiple of $4\ (1 - x^2)$, $8\ (1 - x)$, $8\ (1 + x)$, and $4\ (1 + x^2)$. Ans. $8\ (1 - x^4)$.

CASE II.

114. When the quantities cannot readily be factored by inspection.

1. Find the least common multiple of $a^4 - 5\,a^3 + 9\,a^2 - 7\,a + 2$ and $a^4 - 6\,a^2 + 8\,a - 3$.

OPERATION.

$$\frac{(a^4-5a^3+9a^2-7a+2)(a^4-6a^2+8a-3)}{a^3-3a^2+3a-1} = (a-2)(a^4-6a^2+8a-3)$$

Finding the greatest common divisor of the quantities, $a^3 - 3a^2 + 3a - 1$, we divide their product by it (Art. 112), and have $(a - 2)(a^4 - 6a^2 + 8a - 3)$, the least common multiple required. Hence the

RULE.

Divide the product of the quantities by their greatest common divisor, and the quotient will be their least common multiple.

When more than two quantities are given, find the least common multiple of two of them, and then of that common multiple and the third quantity, and so on.

EXAMPLES.

2. Find the least common multiple of $x^3 - 2x^2y - xy^2 + 2y^3$ and $3x^3 - x^2y - 3xy^2 + y^3$.

Ans. $(x - 2y)(3x - y)(x^2 - y^2)$.

3. Find the least common multiple of $x^3 - 9x^2 + 23x - 15$ and $x^2 - 8x + 7$.

Ans. $(x^3 - 9x^2 + 23x - 15)(x - 7)$.

4. Find the least common multiple of $6x^2 - x - 1$ and $2x^2 + 3x - 2$. Ans. $(2x^2 + 3x - 2)(3x + 1)$.

5. Find the least common multiple of $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

Ans. $x^4 + 6x^3 + 3x^2 - 26x - 24$.

6. Find the least common multiple of $x^2 - 4a^2$, $x^3 + 2ax^2 + 4a^2x + 8a^3$, and $x^3 - 2ax^2 + 4a^2x - 8a^3$.

Ans. $x^4 - 16a^4$.

FRACTIONS.

DEFINITIONS AND NOTATION.

115. A FRACTIONAL UNIT is one of the equal parts into which a unit has been divided. Thus, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{b}$ are fractional units.

116. A FRACTION is a fractional unit, or a collection of fractional units. Thus, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{a}{b}$ are fractions.

The *denominator* of a fraction indicates into how many parts the unit has been divided, and the *numerator* how many of those parts are taken. Thus, $\frac{a}{b}$ indicates that 1 has been divided into b equal parts, and that a such parts are taken.

The numerator and denominator are called the *terms* of a fraction.

117. An ENTIRE QUANTITY is one which has no fractional part; as, $a\,b$, or $a - b$.

118. A MIXED QUANTITY is one having both entire and fractional parts; as, $a - \frac{b}{c}$, or $c + \frac{a}{x+y}$.

119. The VALUE of a fraction is the quotient of its numerator divided by its denominator.

For, a fraction is an expression of division, the numerator answering to the dividend, and the denominator to the divisor. (Art. 18.) Thus, the value of the fraction $\frac{a\,b}{b}$ is a.

GENERAL PRINCIPLES OF FRACTIONS.

120. *If the numerator be multiplied, or the denominator divided, by any quantity, the fraction is multiplied by the same quantity.*

For, let $\frac{a\,b}{b}$ denote any fraction; then,

$$\frac{a\,b}{b} = a.$$

Now, if we multiply its numerator by any quantity b, we have

$$\frac{a\,b^2}{b} = a\,b,$$

and, in like manner, if we divide its denominator by b, we obtain also $a\,b$. Hence, in both cases the value of the fraction has been multiplied by b.

121. *If the denominator be multiplied, or the numerator divided, by any quantity, the fraction is divided by the same quantity.*

For, let $\frac{a\,b^2}{b}$ denote any fraction; then,

$$\frac{a\,b^2}{b} = a\,b.$$

Now, if we multiply its denominator by any quantity b, we have

$$\frac{a\,b^2}{b^2} = a;$$

and, in like manner, if we divide its numerator by b, we obtain also a. Hence, in both cases the value of the fraction has been divided by b.

122. *If the numerator and denominator be both multiplied, or both divided, by the same quantity, the value of the fraction will not be changed.*

For, if any quantity be both multiplied and divided by the same quantity, the value of the former will not be changed. (Art. 46, Ax. 6.)

123. *A fraction is* POSITIVE *when its numerator and denominator have the* SAME *sign, and* NEGATIVE *when they have* DIFFERENT *signs.*

For a fraction represents the quotient of its numerator divided by its denominator; consequently its proper sign must be determined as in division. (Art. 67.)

124. The SIGN of a fraction, or that prefixed to its dividing line, shows whether the fraction is to be added or subtracted.

Thus, in $x + \frac{-a}{b}$, the sign $+$ denotes that $\frac{-a}{b}$, although essentially *negative* (Art. 123), is to be added to x.

The sign written before the dividing line has been termed the *apparent sign* of the fraction, and that depending upon the value expressed by the fraction itself has been termed the *real sign.*

Thus, in $+\frac{-a}{b}$, the apparent sign is $+$, and the real sign $-$.

125. *If any one of the signs prefixed to the numerator, denominator, and dividing line of a fraction be changed, the value of the fraction will be changed accordingly.*

For, $\frac{ab}{b} = a$; $\frac{-ab}{b} = -a$; $\frac{ab}{-b} = -a$; $-\frac{ab}{b} = -a$.

Also,

$$-\frac{-ab}{-b} = -a; \quad -\frac{ab}{-b} = +a; \quad -\frac{-ab}{b} = +a; \quad \frac{-ab}{-b} = a.$$

126. *Any two of the signs prefixed to the numerator, denominator, and dividing line of a fraction may be changed, without affecting the value of the fraction.*

For, $\frac{ab}{b} = -\frac{ab}{-b} = \frac{-ab}{-b} = -\frac{-ab}{b} = +a.$

Also, $-\frac{ab}{b} = \frac{-ab}{b} = \frac{ab}{-b} = -\frac{-ab}{-b} = -a.$

127. *If all the signs prefixed to the terms and the dividing line of a fraction be changed, the value of the fraction will be changed accordingly.*

For, $\frac{ab}{b} = a$, but $-\frac{-ab}{-b} = -a$;

$\frac{-ab}{b} = -a$, but $-\frac{ab}{-b} = a$;

$\frac{ab}{-b} = -a$, but $-\frac{-ab}{b} = a$;

$-\frac{ab}{b} = -a$, but $\frac{-ab}{-b} = a.$

REDUCTION OF FRACTIONS.

128. Reduction of Fractions is the process of changing their forms without altering their values.

CASE I.

129. To reduce a fraction to its lowest terms.

A fraction is in its *lowest terms*, when its terms are prime to each other.

Since dividing both numerator and denominator by the same quantity, or canceling equal factors in each, does not alter the value of the fraction (Art. 122), we have the following

RULE.

Resolve both terms of the fraction into their prime factors, and cancel all that are common to both. Or,

Divide both terms by their greatest common divisor.

Examples.

1. Reduce $\frac{a^2x - x^3}{a^4 - x^4}$ to its lowest terms.

$$\frac{a^2x - x^3}{a^4 - x^4} = \frac{x(a+x)(a-x)}{(a+x)(a-x)(a^2+x^2)} = \frac{x}{a^2+x^2}, \text{ Ans.}$$

2. Reduce $\frac{x^3 - b^2x}{x^2 + 2bx + b^2}$ to its lowest terms.

$$\frac{(x^3 - b^2x) \div (x+b)}{(x^2 + 2bx + b^2) \div (x+b)} = \frac{x^2 - bx}{x+b}, \text{ Ans.}$$

Here the reduction is performed by using the greatest common divisor of the terms of the fraction.

3. Reduce $\frac{56\,b\,m^2x^5}{76\,b\,m\,x^2}$ to its lowest terms. Ans. $\frac{14\,m\,x^4}{19}$.

4. Reduce $\frac{19\,a\,b^2\,c\,d\,e^3}{76\,a^3\,b\,c\,d\,e^5}$ to its lowest terms. Ans. $\frac{b}{4\,a^2\,e^2}$.

5. Reduce $\frac{6\,a^2 + 5\,a\,x - 6\,x^2}{6\,a^2 + 13\,a\,x + 6\,x^2}$ to its lowest terms.

Ans. $\frac{3\,a - 2\,x}{3\,a + 2\,x}$.

6. Reduce $\frac{a^2 - x^2}{a^4 - x^4}$ to its lowest terms. Ans. $\frac{1}{a^2 + x^2}$.

7. It is required to reduce $\frac{x^6 - y^6}{x^4 - y^4}$ to its lowest terms.

Ans. $\frac{x^4 + x^2y^2 + y^4}{x^2 + y^2}$.

8. Reduce $\frac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$ to its lowest terms. Ans. $\frac{2}{3}$.

9. Reduce $\frac{a^3 + (a+b)\,a\,x + b\,x^2}{a^4 - b^2\,x^2}$ to its lowest terms.

Ans. $\frac{a+x}{a^2 - b\,x}$.

10. Reduce $\frac{a\,x^m - b\,x^{m+1}}{a^2 b\,x - b^2 x^3}$ to its lowest terms.

Ans. $\frac{x^{m-1}}{b\,(a+b\,x)}$.

11. Reduce $\frac{b\,x+2}{2\,b + (b^2-4)\,x - 2\,b\,x^2}$ to its lowest terms.

Ans. $\frac{1}{b-2\,x}$.

CASE II.

130. To reduce a fraction to an entire or mixed quantity.

Since the numerator is a dividend, and the denominator a divisor (Art. 119), this is simply a case in division. Hence the

RULE.

Divide the numerator by the denominator, and the quotient will be the entire or mixed quantity.

Examples.

1. Reduce $\frac{a\,b - a^2}{b}$ to a mixed quantity.

$$\frac{a\,b - a^2}{b} = (a\,b - a^2) \div b = a - \frac{a^2}{b}, \text{ Ans.}$$

2. Reduce $\frac{x^3 + y^3}{x+y}$ to an entire quantity.

Ans. $x^2 - x\,y + y^2$.

3. Reduce $\frac{a\,x + a^2 x^2}{a\,x}$ to an entire quantity.

4. Reduce $\frac{10x^2 - 5x + 3}{5x}$ to a mixed quantity.

Ans. $2x - 1 + \frac{3}{5x}$.

5. Reduce $\frac{a^3 + b^3 - x^3}{a - x}$ to a mixed quantity.

Ans. $a^2 + ax + x^2 + \frac{b^3}{a - x}$.

6. Reduce $\frac{2x^2 + 5}{x - 3}$ to a mixed quantity.

Ans. $2x + 6 + \frac{23}{x - 3}$.

7. Reduce $\frac{x^3 - 1}{x - 1}$ to an entire quantity.

Ans. $x^2 + x + 1$.

8. Reduce $\frac{4x^2 - 2x}{2x^3 - x + 1}$ to a mixed quantity.

Ans. $2 - \frac{2}{2x^2 - x + 1}$.

9. Reduce $\frac{m^2(m^4 - n^4) + 3(m^3 - mn^2)}{m^2(m^2 - n^2)}$.

Ans. $m^2 + n^2 + \frac{3}{m}$.

131. By means of *negative exponents*, the value of any fraction whatever may be expressed in the form of an entire quantity.

For, since any fraction is an expression of division,

Any factor may be transferred from either term of a fraction to the other by changing the sign of its exponent. (Art. 77.)

10. Reduce $\frac{15abx^2}{5ab^2m^2}$ to the form of an entire quantity.

$$\frac{15abx^2}{5ab^2m^2} = \frac{3x^2}{bm^2} = 3b^{-1}m^{-2}x^2, \text{ Ans.}$$

11. Change $\frac{7x^2y^3}{4x^3z^2}$ to the form of an entire quantity.

Ans. $7(4^{-1}x^{-1}y^3z^{-2})$.

12. Change $\frac{x^3 y^2}{a^{-2} b^{-1} x^{-1} y}$ to an entire quantity.

Ans. $a^2 b x^4 y$.

13. In $\frac{1}{x^{-3} y^{-2} z^{-1}}$, remove the negative exponents.

14. Reduce $\frac{x^2 (a + b)}{(a - b)^{-2}}$ to an entire quantity.

Ans. $x^2 (a^3 - a^2 b - a b^2 + b^3)$.

15. Reduce $\frac{5 x (x - 1)^{-1} + 3 (x - 1)^{-1}}{(x - 1)^{-2}}$ to an entire quantity. Ans. $5 x^2 - 2 x - 3$.

CASE III.

132. To reduce a mixed quantity to a fractional form.

This is the converse of Case II. Hence the following

RULE.

Multiply the entire part by the denominator of the fraction; add the numerator to the product when the sign of the fraction is plus, and subtract it when the sign is minus, and write the result over the denominator.

Examples.

1. Reduce $a + b - \frac{a^2 - b^2 - 5}{a - b}$ to a fractional form.

$$a + b - \frac{a^2 - b^2 - 5}{a - b} = \frac{a^2 - b^2 - (a^2 - b^2 - 5)}{a - b}$$

$$= \frac{a^2 - b^2 - a^2 + b^2 + 5}{a - b} = \frac{5}{a - b}, \text{ Ans.}$$

2. Change $a + \frac{b^2 - cd}{n}$ to a fractional form.

Ans. $\frac{an + b^2 - cd}{n}$.

3. Reduce $7x - \frac{4n^2 + 5a}{8}$ to a fractional form.

Ans. $\frac{56x - 4n^2 - 5a}{8}$.

4. Reduce $x - \frac{a - b}{m}$ to the form of a fraction.

5. Reduce $a + b - \frac{a^2 + b^2}{a + b}$ to the form of a fraction.

Ans. $\frac{2ab}{a + b}$.

6. Reduce $1 + \frac{b^2 + c^2 - a^2}{2bc}$ to the form of a fraction.

Ans. $\frac{(b + c)^2 - a^2}{2bc}$.

7. Reduce $2a - \frac{3a^2 - 2b^2}{2a}$ to the form of a fraction.

Ans. $\frac{a^2 + 2b^2}{2a}$.

8. Reduce $x + 1 + \frac{x + 1}{x}$ to a fractional form.

Ans. $\frac{(x + 1)^2}{x}$.

9. Reduce $a - b - \frac{a^2 + b^2}{a + b}$ to a fractional form.

Ans. $-\frac{2b^2}{a + b}$.

10. Reduce $x^2 - 3x - \frac{3x(3 - x)}{x - 2}$ to a fractional form.

Ans. $\frac{x(x^2 - 2x - 3)}{x - 2}$.

CASE IV.

133. To reduce fractions to equivalent ones having a common denominator.

Fractions have a common denominator when they have the same quantity for a denominator.

1. Let it be required to reduce $\frac{a}{mx}$ and $\frac{b}{my}$ to fractions having a common denominator.

FIRST OPERATION.

$$\frac{a}{mx} = \frac{a \times my}{mx \times my} = \frac{amy}{m^2xy}.$$

$$\frac{b}{my} = \frac{b \times mx}{my \times mx} = \frac{bmx}{m^2xy}.$$

Multiplying both terms of each fraction by the denominator of the other, which does not change the value expressed (Art. 122), we have fractions equivalent to those given, and with a common denominator, m^2xy.

But this is not the least common denominator, for all the terms of the fractions found have a common factor, m.

Now, the *common* denominator m^2xy is a *common multiple* of the lowest denominators; hence, their *least common* multiple is their *least common* denominator.

SECOND OPERATION.

$$mxy \div mx = y; \quad \frac{a \times y}{mx \times y} = \frac{ay}{mxy}.$$

$$mxy \div my = x; \quad \frac{b \times x}{my \times x} = \frac{bx}{mxy}.$$

The least common multiple of the denominators is mxy. Dividing this by each of those denominators, we ascertain that the multipliers required to change the given fractions to others having the least common denominator are, respectively, y and x. Multiplying both terms of the first fraction by y, and both terms of the second by x, we have fractions equivalent to those given, and with the least common denominator.

RULE.

Multiply each numerator by all the denominators except its own, for new numerators, and all the denominators together for a COMMON *denominator.* Or,

Find the least common multiple of all the denominators for the LEAST COMMON *denominator. Divide this multiple by each denominator, separately, and multiply the corresponding numerators by the quotients for new numerators.*

Before applying the rule, each fraction should be in its lowest terms, and entire and mixed quantities should be changed to a fractional form.

EXAMPLES.

2. Reduce $\frac{3x}{y}$, $\frac{4m}{x}$, and $\frac{a}{b-c}$ to fractions having a common denominator.

Ans. $\frac{3bx^2 - 3cx^2}{bxy - cxy}$, $\frac{4bmy - 4cmy}{bxy - cxy}$, $\frac{axy}{bxy - cxy}$.

3. Reduce a, b, c, d, and $\frac{a}{b}$ to fractions having a common denominator. Ans. $\frac{ab}{b}$, $\frac{b^2}{b}$, $\frac{bc}{b}$, $\frac{bd}{b}$, $\frac{a}{b}$.

4. Reduce $\frac{2x+1}{6a}$ and $\frac{5x}{4}$ to fractions having the least common denominator. Ans. $\frac{2(2x+1)}{12a}$, $\frac{15ax}{12a}$.

5. Reduce $\frac{-5y}{7bx}$, $\frac{3c}{d}$, $\frac{4}{-5}$ to fractions having a common denominator.

Ans. $\frac{25dy}{-35bdx}$, $\frac{-105bcx}{-35bdx}$, $\frac{28bdx}{-35bdx}$.

6. Reduce $\frac{a^2}{a+b}$, $\frac{a+b}{a-b}$, and $\frac{a^2b^2}{a^2-b^2}$ to fractions having the least common denominator.

Ans. $\frac{a^2(a-b)}{a^2-b^2}$, $\frac{(a+b)^2}{a^2-b^2}$, $\frac{a^2b^2}{a^2-b^2}$.

7. Reduce $\frac{a\,y}{1-x}$, $\frac{a\,x^2}{(1-x)^2}$, and $\frac{x\,y^3}{(1-x)^3}$ to fractions having the least common denominator.

Ans. $\frac{a\,y\,(1-x)^2}{(1-x)^3}$, $\frac{a\,x^2\,(1-x)}{(1-x)^3}$, $\frac{x\,y^3}{(1-x)^3}$.

134. A fraction may be reduced to an equivalent one having a *given* denominator, by *dividing the given denominator by the denominator of the fraction, and multiplying both terms by the quotient.*

8. Change $\frac{a\,x^2}{y}$ into a fraction having $b\,y$ for a denominator.

$$b\,y \div y = b\,;\quad \frac{a\,x^2 \times b}{y \times b} = \frac{a\,b\,x^2}{b\,y},\ \text{Ans.}$$

9. Change $\frac{a-b}{a^2-a\,b+b^2}$ into a fraction having a^3+b^3 for a denominator. Ans. $\frac{a^3-b^3}{a^3+b^3}$.

10. Change $\frac{2\,d}{5}$ and $\frac{3\,c^2}{a\,b}$ into fractions having $10\,a^2\,b$ for a common denominator. Ans. $\frac{4\,a^2\,b\,d}{10\,a^2\,b}$, $\frac{30\,a\,c^2}{10\,a^2\,b}$.

11. Convert $a\,b$, $\frac{a-b}{a+b}$, and $\frac{a+b}{a-b}$ into fractions having a^2-b^2 for a common denominator.

Ans. $\frac{a\,b\,(a^2-b^2)}{a^2-b^2}$, $\frac{(a-b)^2}{a^2-b^2}$, $\frac{(a+b)^2}{a^2-b^2}$.

ADDITION OF FRACTIONS.

135. When fractions have a common denominator, they may be added by finding the sum of their numerators.

For the sum of $\frac{a}{c}$ and $\frac{b}{c}$ is evidently the same as a divided by c, added to b divided by c, or the sum of a and b divided by c.

Also, since $\frac{a}{c} = a\,c^{-1}$, and $\frac{b}{c} = b\,c^{-1}$ (Art. 131),

$$\frac{a}{c} + \frac{b}{c} = a\,c^{-1} + b\,c^{-1} = (a+b)\,c^{-1} = \frac{a+b}{c}.$$

Hence the

RULE.

Reduce the fractions, if necessary, to equivalent ones having a common denominator. Add the numerators, and write the sum over the common denominator.

The final result should be reduced to its lowest terms, whenever such reduction is possible.

EXAMPLES.

1. Find the sum of $\frac{x}{x-3}$ and $\frac{x}{x+3}$.

$$\frac{x}{x-3} + \frac{x}{x+3} = \frac{x^2+3x}{x^2-9} + \frac{x^2-3x}{x^2-9}$$

$$= \frac{x^2+3x+x^2-3x}{x^2-9} = \frac{2x^2}{x^2-9}, \text{ Ans.}$$

2. Find the sum of $\frac{4a}{7}$ and $\frac{a-3}{4}$. Ans. $\frac{23a-21}{28}$.

3. Find the sum of $4m$, $\frac{3a-1}{2}$, and $\frac{4n+2}{3}$.

Ans. $\frac{9a+24m+8n+1}{6}$.

4. Find the sum of $\frac{4a-3}{5}$, $\frac{7a+1}{3}$, and $\frac{3a}{2}$.

5. Find the sum of $\frac{3}{a+b}$ and $\frac{3}{a-b}$. Ans. $\frac{6a}{a^2-b^2}$.

6. Add $a - \frac{3x^2}{b}$ to $b + \frac{2ax}{c}$.

Ans. $a + b + \frac{2abx - 3cx^2}{bc}$.

7. Add $2+\frac{a+b}{a-b}$ to $\frac{a-b}{a+b}$. Ans. $\frac{4a^2}{a^2-b^2}$.

8. Add $\frac{1+x}{1+x+x^2}$ to $\frac{1-x}{1-x+x^2}$. Ans. $\frac{2}{1+x^2+x^4}$.

9. Find the sum of $\frac{3}{a}$, $-\frac{a+1}{b-1}$, and $-\frac{a-1}{b+1}$.

Ans. $\frac{3b^2-2a^2b-2a-3}{ab^2-a}$.

10. Find the sum of $2a$, $3a+\frac{2x}{5}$, and $a-\frac{8x}{9}$.

Ans. $6a-\frac{22x}{45}$.

11. Add $\frac{a+b}{(b-c)(c-a)}$, $\frac{b+c}{(c-a)(a-b)}$, and $\frac{c+a}{(a-b)(b-c)}$.

Ans. 0.

12. Add $\frac{1}{a(a-b)(a-c)}$, $\frac{1}{b(b-c)(b-a)}$, and $\frac{1}{c(c-a)(c-b)}$. Ans. $\frac{1}{abc}$.

SUBTRACTION OF FRACTIONS.

136. When two fractions have a common denominator, the one may be subtracted from the other by finding the difference of their numerators.

For $\frac{b}{c}$ subtracted from $\frac{a}{c}$ is evidently the same as b divided by c, subtracted from a divided by c, or a minus b divided by c.

Also, since $\frac{a}{c}=ac^{-1}$, and $\frac{b}{c}=bc^{-1}$,

$$\frac{a}{c}-\frac{b}{c}=ac^{-1}-bc^{-1}=(a-b)c^{-1}=\frac{a-b}{c}.$$

Hence the

RULE.

Reduce the fractions, if necessary, to equivalent ones having a common denominator. Subtract the numerator of the subtrahend from that of the minuend, and write the difference over the common denominator.

EXAMPLES.

1. From $\frac{a}{b}$ subtract $\frac{c}{d}$.

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}, \text{ Ans.}$$

2. From $\frac{7x}{5}$ subtract $\frac{4x}{7}$. Ans. $\frac{29x}{35}$.

3. From $\frac{2}{a-1}$ subtract $\frac{2}{a+1}$. Ans. $\frac{4}{a^2-1}$.

4. Subtract $\frac{2a-4b}{5b}$ from $\frac{3a-2b}{3c}$.

5. Subtract $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$. Ans. $\frac{4xy}{x^2-y^2}$.

6. Subtract $a - \frac{a-b}{d}$ from $3a + \frac{a+b}{d}$.

Ans. $2a + \frac{2a}{d}$.

7. Subtract $x - \frac{4a-b}{2}$ from $7x - \frac{3a-2b}{3}$.

Ans. $6x + a + \frac{b}{6}$.

8. From $\frac{(a+b)^2}{ab}$ take $\frac{(a-b)^2}{ab}$. Ans. 4.

9. From $\frac{(a^2+b^2)^2}{ab\,(a-b)^2}$ take $\frac{a}{b} + \frac{b}{a} + 2$. Ans. $\frac{4ab}{(a-b)^2}$

10. From $\frac{5}{2\,(x+1)} - \frac{1}{10\,(x-1)}$ take $\frac{24}{5\,(2x+3)}$.

Ans. $\frac{2x-3}{(x^2-1)\,(2x+3)}$.

MULTIPLICATION OF FRACTIONS.

137. It has been shown that a fraction may be multiplied by an entire quantity, either by multiplying its numerator, or by dividing its denominator (Art. 120).

If, however, both multiplicand and multiplier are fractions, their product equals the product of their numerators divided by the product of their denominators.

For, let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

Then, since $\frac{a}{b} = a\,b^{-1}$, and $\frac{c}{d} = c\,d^{-1}$,

$$\frac{a}{b} \times \frac{c}{d} = a\,b^{-1} \times c\,d^{-1} = a\,c\,b^{-1}\,d^{-1} = \frac{a\,c}{b\,d}.$$

Hence the

RULE.

Reduce entire and mixed quantities, if any, to fractional forms. Multiply the numerators together for a new numerator, and the denominators for a new denominator.

When there are common factors in the numerators and denominators, they may be canceled before performing the multiplication.

EXAMPLES.

1. Multiply $\frac{3x^2 - x}{5}$ by $\frac{10}{2x^2 - 4x}$.

$$\frac{3x^2-x}{5} \times \frac{10}{2x^2-4x} = \frac{3x^2-x}{5} \times \frac{5}{x^2-2x} = \frac{3x-1}{1} \times \frac{1}{x-2} = \frac{3x-1}{x-2}, \text{ Ans.}$$

2. Multiply $\frac{a^2\,b\,c}{m\,n^2}$ by $\frac{a^3\,b^2}{m^3\,n\,d}$. Ans. $\frac{a^5\,b^3\,c}{m^4\,n^3\,d}$.

3. Multiply $\frac{3\,a^3\,x}{7\,h^4}$ by $\frac{4\,a\,b}{5\,h\,m}$. Ans. $\frac{12\,a^4\,b\,x}{35\,h^5\,m}$.

4. Required the product of $\frac{3\,a\,b\,x^2}{5\,a\,y^2}$ by $\frac{5\,x\,y^2}{3\,a\,b\,x^3}$.

5. Multiply $\frac{m+n}{x+y}$ by $\frac{m+n}{x+y}$. Ans. $\frac{m^2+2mn+n^2}{x^2+2xy+y^2}$.

6. Multiply $a+\frac{hy}{a}$ by $\frac{m}{n}$. Ans. $\frac{a^2m+hmy}{an}$.

7. Multiply $\frac{my^n}{4ax}$ by $\frac{ax}{my^n}$. Ans. $\frac{1}{4}$.

8. Multiply $\frac{4x+2}{3}$ by $\frac{5x}{2x+1}$. Ans. $\frac{10x}{3}$.

9. Multiply $\frac{2x-1}{x+2}$ by $\frac{3x+1}{x-3}$. Ans. $\frac{6x^2-x-1}{x^2-x-6}$.

10. Multiply $\frac{x^n}{y^m}$ by $\frac{x^m}{y^n}$. Ans. $\frac{x^{m+n}}{y^{m+n}}$.

11. Multiply $\frac{(a-b)^2}{a+b}$ by $\frac{b}{x(a-b)}$. Ans. $\frac{b(a-b)}{x(a+b)}$.

12. Multiply x^2-x+1 by $\frac{1}{x^3}+\frac{1}{x}+1$.

Ans. $x^2+1+\frac{1}{x^2}$.

13. Find the product of $\frac{x^n}{y^m}$ by $\frac{y^n}{x^m}$. Ans. $\frac{x^{n-m}}{y^{m-n}}$.

14. Multiply together $\frac{1-x^2}{1+y}$, $\frac{1-y^2}{x+x^2}$, and $1+\frac{x}{1-x}$.

Ans. $\frac{1-y}{x}$.

15. Multiply $\frac{3ab^2}{a-b}$ by a^2-b^2. Ans. $3ab^2(a+b)$.

16. Required the value of $b\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$.

Ans. $\frac{a^4-b^4}{a^2b}$.

DIVISION OF FRACTIONS.

138. It has been shown that a fraction may be divided by an entire quantity, either by dividing its numerator or by multiplying its denominator (Art. 121).

If, however, both dividend and divisor are fractions, their quotient equals the product of the dividend by the reciprocal of the divisor, or by the divisor inverted (Art. 43).

For, let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

Then, since $\frac{a}{b} = a\,b^{-1}$, and $\frac{c}{d} = c\,d^{-1}$,

$$\frac{a}{b} \div \frac{c}{d} = a\,b^{-1} \div c\,d^{-1} = \frac{a\,b^{-1}}{c\,d^{-1}} = \frac{a\,d}{b\,c}; \text{ but}$$

$$\frac{a\,d}{b\,c} = \frac{a}{b} \times \frac{d}{c}.$$

Hence the

RULE.

Reduce entire and mixed quantities, if any, to fractional forms. Invert the terms of the divisor, and proceed as in multiplication.

When there are common factors in the numerators and denominators after the divisor is inverted, they may be canceled.

EXAMPLES.

1. Find the quotient of $\frac{x^2-9}{5}$ divided by $\frac{x+3}{4}$.

$$\frac{x^2-9}{5} \div \frac{x+3}{4} = \frac{x^2-9}{5} \times \frac{4}{x+3} = \frac{x-3}{5} \times \frac{4}{1} = \frac{4x-12}{5}, \text{ Ans.}$$

2. Divide $\frac{7m^2}{2}$ by $\frac{3n^2}{13}$. Ans. $\frac{91m^2}{6n^2}$.

3. Divide $\frac{1}{xy}$ by xy. Ans. $\frac{1}{x^2y^2}$.

4. Divide $\frac{3x+1}{9}$ by $\frac{4x}{3}$. Ans. $\frac{3x+1}{12x}$.

5. Divide $\frac{4x}{2x-1}$ by $\frac{2}{x+1}$.

6. Divide $\frac{2a+b}{3a-2b}$ by $\frac{3a+2b}{4a+b}$. Ans. $\frac{8a^2+6ab+b^2}{9a^2-4b^2}$.

7. Divide $\frac{x^2-y^2}{x^2-2xy+y^2}$ by $\frac{x^2+xy}{x-y}$. Ans. $\frac{1}{x}$.

8. Divide $\frac{9x^2-4y^2}{x+y}$ by $\frac{3x-2y}{x^2-y^2}$.

Ans. $3x^2-xy-2y^2$.

9. Divide $\frac{2a-b}{4ac}$ by $\frac{2ac}{3b-1}$.

Ans. $\frac{-2a+b-3b^2+6ab}{8a^2c^2}$.

10. Divide $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a-b}-\frac{b}{a+b}$. Ans. 1.

11. Divide $x^4-\frac{1}{x^4}$ by $x+\frac{1}{x}$.

Ans. $x^3-x+\frac{1}{x}-\frac{1}{x^3}$.

12. Divide $\frac{c-b}{c+b}-\frac{c^2-b^2}{c^2+b^2}$ by $\frac{c+b}{c-b}+\frac{c^2+b^2}{c^2-b^2}$.

Ans. $-\frac{bc(b-c)^2}{b^4+c^4+b^2c^2}$.

139. A Complex Fraction is one having a fraction in its numerator, or denominator, or both. It may be regarded as a case in division, since its numerator answers to the dividend, and its denominator to the divisor.

However, since multiplying a fraction by any multiple of its denominator must cancel that denominator, to simplify a complex fraction, we may *multiply both of its terms by the least common multiple of their denominators.*

13. Reduce $\frac{\frac{a}{c}}{\frac{d}{b}}$ to the simplest form.

$$\frac{\frac{a}{c}}{\frac{d}{b}}=\frac{a}{c}\times\frac{b}{d}=\frac{ab}{cd};\text{ or }\frac{\frac{a}{c}}{\frac{d}{b}}=\frac{\frac{a}{c}\times bc}{\frac{d}{b}\times bc}=\frac{ab}{cd},\text{ Ans.}$$

Here, it will be seen that proceeding as in division, and multiplying by the least common multiple of the denominators of the terms, produce the same result.

14. Reduce $\dfrac{\frac{a}{b}}{m+n}$ to a simple fraction. Ans. $\dfrac{a}{bm+bn}$.

15. Reduce $\dfrac{a+\frac{b}{c}}{x-\frac{m}{n}}$ to a simple fraction. Ans. $\dfrac{acn+bn}{cnx-cm}$.

16. Reduce $\dfrac{m-\frac{n}{3}}{x}$ to a simple fraction. Ans. $\dfrac{3m-n}{3x}$.

17. Simplify $\dfrac{y-x+\frac{a}{2}}{7\frac{3}{4}}$. Ans. $\dfrac{4y-4x+2a}{31}$.

18. Simplify $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$. Ans. $\dfrac{4}{3(x+1)}$.

140. When the terms of a fraction contain negative exponents, the fraction may be regarded as a complex one.

If the letter or quantity which bears the negative exponent is a *factor* of either numerator or denominator, as a whole, the negative exponent may be removed as in Art. 131; otherwise, both numerator and denominator must be multiplied by that letter or quantity with an equal positive exponent, in accordance with Art. 139.

19. Free the fraction $\dfrac{x^2+y^{-3}}{mn^{-4}e^{-2}}$ from negative exponents. Ans. $\dfrac{e^2n^4x^2y^3+e^2n^4}{my^3}$.

20. Free the fraction $\dfrac{1-a^{-2}-y^3}{1-x^{-3}y^{-2}+x^{-2}}$ from negative exponents. Ans. $\dfrac{a^2x^3y^2-x^3y^2-a^2x^3y^4}{a^2x^2y^2-a^2+a^2xy^2}$.

SIMPLE EQUATIONS.

141. An EQUATION is an expression of equality between two quantities. Thus,

$$x + 4 = 16$$

is an equation, expressing the equality of the quantities $x + 4$ and 16.

142. The FIRST MEMBER of an equation is the quantity on the left of the sign of equality.

The SECOND MEMBER is the quantity on the right of the same sign.

The *sides* of an equation are its two members.

143. An IDENTICAL EQUATION is one in which the two members are the same algebraic expression, or become the same on performing the operations indicated; as,

$$x - y = x - y, \text{ or } 2a + 2bc = 2(a + bc).$$

144. A NUMERICAL EQUATION is one in which all the known quantities are expressed by figures; as,

$$2x - x = 17 - 5.$$

145. A LITERAL EQUATION is one in which some or all the known quantities are expressed by letters; as,

$$2x + a = x^2 - 10.$$

146. The DEGREE of an equation containing but one unknown quantity is denoted by the exponent of the highest power of the unknown quantity in the equation. Thus,

An equation of the *first degree* is one having no higher power of the unknown quantity than its *first* power; as,

$$x + 14 = 28 - 4, \text{ or } cx = a^2 + bd.$$

An equation of the *second degree* is one in which the highest power of the unknown quantity is the *second* power, or square ; as,

$$3x^2 - 2x = 65.$$

In like manner, we have equations of the *third degree, fourth degree,* . . . . . n*th degree.*

When an equation contains more than one unknown quantity, its degree is determined by the greatest *sum* of the exponents of the unknown quantities, in any term. Thus,

$x + xy = 25$ is an equation of the second degree.

$x^2 - y^2z = ab^3$ is an equation of the third degree.

147. A SIMPLE EQUATION is an equation of the first degree.

148. The ROOT of an equation is the value of its unknown quantity.

The root is *verified*, or the equation *satisfied*, when, the root being substituted for its symbol, the equation becomes identical.

TRANSFORMATION OF EQUATIONS.

149. The TRANSFORMATION of an equation is the process of changing its form without destroying the equality.

150. The operations required in the transformation are based upon the general principle deduced directly from the axioms (Art. 46) : —

If the same operations are performed upon equal quantities, the results will be equal ; hence,

Both members of an equation may be increased, diminished, multiplied, or divided by the same quantity, without destroying the equality.

CASE I.

151. To transpose terms of an equation.

TRANSPOSITION is the process of changing terms from one member of an equation to the other, without destroying the equality.

1. Let it be required to transpose $-a$ in

$$x - a = b.$$

Adding a to each member (Art. 150); then

$$x - a + a = b + a,$$

or,
$$x = b + a.$$

Where $-a$ has been transposed with the sign changed.

2. Let it be required to transpose a in

$$x + a = b.$$

Subtracting a from each member (Art. 150); then

$$x + a - a = b - a,$$

or,
$$x = b - a.$$

Where a has been transposed with the sign changed.

RULE.

Any term may be transposed from one member of an equation to the other, provided its sign be changed.

If the same term appear in both members of an equation, affected with the same sign, it may be suppressed.

It also follows, since every term may be transposed, that *the* SIGNS *of all the terms of an equation may be changed, without destroying the equality.*

Transpose the unknown terms to the first member, and the known terms to the second, in the following

EXAMPLES.

3. $3x - 2a = 45 + 2x$.

Ans. $3x - 2x = 2a + 45$.

4. $4x + 9 = 25 - 12x$.

5. $4a^2x + b^2 = -4abx + 4ac + b^2$.

Ans. $4a^2x + 4abx = 4ac$.

6. $ac + cx - ad = 2a - 7x$.

Ans. $7x + cx = 2a - ac + ad$.

7. $bc + a^2x - mn^2 = bx + ad - 5$.

Ans. $a^2x - bx = ad - 5 - bc + mn^2$.

CASE II.

152. To clear an equation of fractions.

1. Let it be required to clear of fractions

$$\frac{x}{ac} + \frac{x}{bc} = d.$$

Since multiplying a fraction by a quantity equal to its denominator will cancel the denominator, if both members of the equation be multiplied by ac and by bc, or by $ac \times bc$, the fractions will be removed, without destroying the equality (Art. 150).

But, multiplying by any multiple of the denominators will cause them to disappear; hence, multiplying by their least common multiple, abc, as the least possible multiple that will transform the fractions to the form of entire quantities, we have

$$bx + ax = abcd.$$

Hence the

RULE.

Multiply each term of the equation by all the denominators, or by the least common multiple of the denominators.

When a fraction is preceded by —, since it is to be

subtracted, on removing its denominator, all the signs of its numerator must be changed (Art. 125).

Negative exponents are removed from an equation in the same way as fractions.

Clear the equations of fractions in the following

EXAMPLES.

2. $\frac{x}{2} - \frac{a+x}{3} = 7\frac{1}{2}$. Ans. $3x - 2a - 2x = 45$.

3. $\frac{ax}{b} - c = \frac{dx}{e} - \frac{m}{n}$.

Ans. $aenx - bcen = bdnx - bem$.

4. $\frac{ax+b}{c} - \frac{a}{b} = \frac{cx+d}{bc}$.

5. $\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13}{6}x$.

Ans. $10x - 32x - 312 = 21 - 52x$.

6. $x - \frac{ax}{b} + cxd^{-1} - ae^{-1} = 0$.

Ans. $bdex - adex + bcex - abd = 0$.

7. $21 + \frac{3x-11}{16} = \frac{5(x-1)}{8} - \frac{97-7x}{2}$.

Ans. $336 + 3x - 11 = 10x - 10 - 776 + 56x$.

SOLUTION OF EQUATIONS.

153. The SOLUTION OF AN EQUATION is the process of finding the value, or values, of the unknown quantity, or the roots of the equation.

154. To solve an equation, such transformations are required as shall make the unknown quantity stand alone as one member of the equation, and all the known quantities as the other member.

155. To solve simple equations of one unknown quantity.

1. Let it be required to find the value of x in the equation,

$$\frac{2x}{5} + \frac{x}{2} = 22 + \frac{x}{6}. \quad (1)$$

Clearing (1) of fractions, gives

$$12x + 15x = 660 + 5x. \quad (2)$$

Transposing in (2), we have

$$12x + 15x - 5x = 660. \quad (3)$$

Uniting similar terms in (3),

$$22x = 660. \quad (4)$$

Dividing both members of (4) by the coefficient of x, we obtain

$$x = 30.$$

This value of x we may verify (Art. 148). Thus, substituting the value of x in (1), the original equation becomes

$$\frac{60}{5} + \frac{30}{2} = 22 + \frac{30}{6};$$

or,

$$12 + 15 = 22 + 5;$$

whence

$$27 = 27,$$

in which the members are the same; and the value of x is verified.

Hence the following

RULE.

Clear the equation of fractions, if it has any.

Transpose the unknown terms to the first member, and the known terms to the second member, and reduce each member to its simplest form.

Divide both members by the coefficient of the unknown quantity, and the second member of the resulting equation will be the value of the unknown quantity.

In some cases the operation may be abbreviated by some artifice, such as transposing and uniting similar terms before clearing of fractions, or by partially clearing at first, and then making reductions.

EXAMPLES.

2. Solve the equation,

$$\frac{2x}{3} + 12 = \frac{4x}{5} + 6$$

Clearing of fractions,

$$10x + 180 = 12x + 90$$

Transposing and uniting,

$$-2x = -90$$

Dividing by -2, $\qquad x = 45$

3. Solve the equation,

$$\frac{1 - bx}{a} = \frac{1 - ax}{b}$$

Clearing of fractions, $\quad b - b^2x = a - a^2x$

Transposing, $\quad a^2x - b^2x = a - b$

Factoring, $\quad (a^2 - b^2)\,x = a - b$

Dividing by $a^2 - b^2$, $\quad x = \dfrac{a-b}{a^2-b^2} = \dfrac{1}{a+b}$

4. Solve the equation,

$$5x + \frac{4x+3}{9} + \frac{7x-29}{5x-12} = 5x + \frac{8x+19}{18}$$

Suppressing $5x$ in each member (Art. 151),

$$\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}$$

Multiplying by 18,

$$8x + 6 + \frac{126x - 522}{5x-12} = 8x + 19$$

Subtracting $8x + 6$ from each member (Art. 150),

$$\frac{126x - 522}{5x-12} = 13$$

Clearing of fractions,

$$126x - 522 = 65x - 156$$

Transposing and uniting,

$$61x = 366$$

Dividing by the coefficient of x,

$$x = 6$$

Solve each of the following equations.

5. $12x - 3x - 2x = 63$. Ans. $x = 9$.

6. $x - \frac{x}{4} + 13 = \frac{x}{2} + 40$. Ans. $x = 108$.

7. $7x + 2x = 12x - 36$.

8. $\frac{x}{5} + \frac{x}{12} = \frac{x}{10} + 22$. Ans. $x = 120$.

9. $\frac{a}{x} = b + c$. Ans. $x = \frac{a}{b+c}$.

10. $x - \frac{x}{7} + 20 = \frac{x}{2} + \frac{x}{4} + 26$. Ans. $x = 56$.

11. $3x + \frac{3x}{4} + 15 = \frac{x}{2} + 41$. Ans. $x = 8$.

12. $x - \frac{4x+8}{6} = 8$. Ans. $x = 28$.

13. $21 + \frac{3x-11}{16} = \frac{5x-5}{8} + \frac{97-7x}{2}$.

14. $17x - \frac{5x-4}{3} - \frac{8x+4}{5} = 20x - \frac{3x+8}{2} - 5$.

Ans. $x = 2$.

15. $ax^2 + bx = mx^2 + nx$. Ans. $x = \frac{n-b}{a-m}$.

16. $a + x^{-1} = b + c + dx^{-1}$. Ans. $x = \frac{d-1}{a-b-c}$.

17. $\frac{2(4x+3)}{x+3} + \frac{3}{x+1} = 8$. Ans. $x = -\frac{3}{5}$.

18. $\frac{x}{4} - 4 = \frac{5x}{3} - \frac{7}{6}$. Ans. $x = -2$.

19. $ax + b = \frac{x}{a} + b^{-1}$. Ans. $x = \frac{a(1-b^2)}{b(a^2-1)}$.

20. $1.2x - \frac{.18x - .05}{.5} = .4x + 8.9$. Ans. $x = 20$.

21. $(a+x)(b+x) - a(b+c) = \frac{a^2c}{b} + x^2$.

Ans. $x = \frac{ac}{b}$.

22. $\frac{3x}{b} - \frac{x}{c} = m - c$. Ans. $x = \frac{bc(m-c)}{3c-b}$.

23. $\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}$. Ans. $x = 6$.

24. $\frac{7x+9}{4} - \left(x - \frac{2x-1}{9}\right) = 7$. Ans. $x = 5$.

25. $\frac{(a-b)x}{2} + \frac{x}{3} = \frac{ab}{4} + a$. Ans. $\frac{3a(b+4)}{6(a-b)+4}$.

26. $4.8x - \frac{.72x - .05}{.5} = 1.6x + 8.9$. Ans. $x = 5$.

27. $\frac{a-x}{b} - \frac{4a-x}{c} = a - b$.

Ans. $x = \frac{4ab - ac + abc - b^2c}{b-c}$.

28. $\frac{x}{a} - \frac{x}{b} + \frac{x}{c} = \frac{2}{ac}$. Ans. $x = \frac{2b}{bc - ac + ab}$.

29. $\frac{1}{ab - ax} + \frac{1}{bc - bx} = \frac{1}{ac - ax}$.

Ans. $x = \frac{b(a-b+c)}{a}$.

30. $(x - \frac{5}{2})(x + \frac{3}{2}) - (x-5)(x+3) - \frac{93}{4} = 0$.

Ans. $x = 12$.

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

156. A PROBLEM is a question proposed for solution, or something to be done.

157. The SOLUTION of a Problem by Algebra consists of two distinct parts: —

1. The STATEMENT, or the process of expressing the conditions of the problem in one or more equations.

2. The SOLUTION of the resulting equation or equations, or the process of determining from them the value of the unknown quantities.

158. The statement of a problem often includes a consideration of *ratio* and *proportion* (Art. 23).

RATIO is the relation, in respect to magnitude, which one quantity bears to another of the same kind, and is the result arising from the division of one quantity by the other. PROPORTION is an equality of ratios. Thus,

$a : b$, or $\frac{a}{b}$, indicates the ratio of a to b;

$a : b = c : d$ indicates a proportion.

159. In a proportion, the relation of the terms is such that the product of the first and fourth is equal to the product of the second and third.

For, $a : b = c : d$ is the same as $\frac{a}{b} = \frac{c}{d}$, which gives

$$a d = b c.$$

160. Much must depend upon the skill and ingenuity of the operator, in the statement of problems, as will be seen from the following examples.

1. What number is that to which if four sevenths of itself be added, the sum will equal 99?

SOLUTION.

Let x = the number.

Then $\frac{4x}{7}$ = four sevenths of it.

By the conditions, $x + \frac{4x}{7} = 99$

Clearing of fractions, $7x + 4x = 693$

Or, $11x = 693$

Whence, $x = 63$, the number.

VERIFICATION. $\frac{4}{7}$ of $63 = 36$; and $63 + 36 = 99$.

2. The sum of two numbers is 144, and their difference is 30; required the numbers.

SOLUTION.

Let x = the smaller number.

Then $x + 30$ = the larger number.

By the conditions, $x + x + 30 = 144$

Transposing and uniting, $2x = 114$

Whence, $x = 57$, the smaller number,

and $x + 30 = 87$, the larger number.

VERIFICATION. $57 + 57 + 30 = 144$.

3. A workman engaged for 48 days, at the rate of $2 per day and his board, which is estimated at $1 per day. At the end of the time he receives $42 only, his employer having deducted the cost of the board for every day he was idle. How many days did he work?

SOLUTION.

Let x = number of days he worked,
and $48 - x$ = number of days he was idle.
Then $2x$ = amount received for labor,
and $48 - x$ = amount deducted for board.

By the conditions, $2x - (48 - x) = 42$
Or, $2x - 48 + x = 42$
Transposing and uniting, $3x = 90$
Whence, $x = 30$, number of days he worked.

4. A can do a piece of work in 8 days, which it requires B 10 days to perform; in how many days can it be done by both working together?

SOLUTION.

Let x = number of days required.
Then $\frac{1}{x}$ = what both can do in one day.
Also, $\frac{1}{8}$ = what A can do in one day,
and $\frac{1}{10}$ = what B can do in one day.

By the conditions, $\frac{1}{8} + \frac{1}{10} = \frac{1}{x}$
Clearing of fractions, $5x + 4x = 40$
Or, $9x = 40$
Whence, $x = 4\frac{4}{9}$, no. of days required.

VERIFICATION. $\frac{1}{8} + \frac{1}{10} = \frac{9}{40}$.

5. Two pieces of cloth were purchased at the same price per yard, but as they were of different lengths, the one cost \$ 5 and the other \$ 6.50. If each had been 10 yards longer, their lengths would have been as 5 to 6. Required the length of each piece.

SOLUTION.

Since the price of each per yard is the same, the lengths of the two pieces must be in the ratio of their prices, that is, as 5 to $6\frac{1}{2}$, or 10 to 13. Therefore,

Let $10x$ = length of first piece, in yards,
and $13x$ = length of second piece, in yards.

By the conditions, $10x + 10 : 13x + 10 :: 5 : 6$
Or, by Art. 159, $60x + 60 = 65x + 50$
Transposing and uniting, $-5x = -10$
Whence, $x = 2$
Then $10x = 20$, length of first,
and $13x = 26$, length of second.

VERIFICATION. $20 + 10 : 26 + 10 = 5 : 6$.

6. A general arranging his troops in the form of a solid square, finds he has 21 men over, but attempting to add 1 man to each side of the square, finds he wants 200 men to fill up the square; required the number of men on a side at first, and the whole number of troops.

SOLUTION.

Let x = the number on a side at first.
Then $x^2 + 21$ = the whole number of troops.

By the conditions, $(x+1)^2 = x^2 + 21 + 200$
Or, $x^2 + 2x + 1 = x^2 + 221$
Transposing and uniting, $2x = 220$
Whence, $x = 110$, no. men on a side at first.
Squaring, $x^2 = 12100$
and $x^2 + 21 = 12121$, the whole number.

From the preceding examples and illustrations is deduced the following

RULE.

Denote the unknown quantity or quantities by some of the final letters of the alphabet.

Form an equation, by indicating the operations required to verify the answer, were it already obtained.

Determine the value of the unknown quantity in the equation thus formed.

PROBLEMS.

7. My horse and chaise are worth $336, but the horse is worth two times as much as the chaise. Required the value of each. Ans. Horse, $224; chaise, $112.

8. A drover has a lot of oxen and cows, for which he gave $1428. For the oxen he gave $55 each, and for the cows $32 each, and he had twice as many cows as oxen. Required the number of each. Ans. 12 oxen; 24 cows.

9. What two numbers are those whose difference is 3, and the difference of whose squares is 51? Ans. 7 and 10.

10. A gentleman, at his decease, left an estate of $1872 for his wife, three sons, and two daughters. His wife was to receive three times as much as either of her daughters, and each son to receive one half as much as each of the daughters. Required the sum that each received.

Ans. Wife, $864; daughters, $288 each; sons, $144 each.

11. A laborer agreed to serve for 36 days on these conditions, that for every day he worked he was to receive $1.25, but for every day he was absent he was to forfeit $0.50. At the end of the time he received $17. It is required to find how many days he labored, and how many days he was absent.

Ans. He labored 20 days, and was absent 16 days.

12. A man being asked the value of his horse and saddle, replied that his horse was worth $114 more than his

saddle, and that $\frac{2}{3}$ the value of the horse was 7 times the value of his saddle. What was the value of each?

Ans. The saddle, $12; the horse, $126.

13. What number is that from which if 7 be subtracted, $\frac{1}{6}$ of the remainder will be 5? Ans. 37.

14. In a garrison of 2744 men, there are 2 cavalry soldiers to 25 infantry, and half as many artillery as cavalry. Required the number of each.

Ans. Infantry, 2450; cavalry, 196; artillery, 98.

15. The stones which pave a square court would just cover a rectangular area, whose length is 6 yards longer, and breadth 4 yards shorter, than the side of the square. Find the area of the court. Ans. 144 square yards.

16. A person has traveled altogether 3036 miles, of which he has gone 7 miles by water to 4 on foot, and 5 by water to 2 on horseback. How many miles did he travel in each manner?

Ans. By water, 1540; on foot, 880; horseback, 616.

17. A certain man added to his estate $\frac{1}{4}$ its value, and then lost $760; but afterwards having gained $600, his property then amounted to $2000. What was the value of his estate at first? Ans. $1728.

18. A capitalist invested $\frac{2}{3}$ of a certain sum of money in government bonds paying 5 per cent. interest, and the remainder in bonds paying 6 per cent.; and found the interest of the whole per annum to be $180. Required the amount of each kind of bonds.

Ans. 5 per cent. bonds, $1200; 6 per cent., $2000.

19. A woman sells half an egg more than half her eggs. Again she sells half an egg more than half her remaining eggs. A third time she does the same; and now she has sold all her eggs. How many had she at first? Ans. 7.

20. A merchant has grain worth 9 shillings per bushel, and other grain worth 13 shillings per bushel. In what

proportion must he mix 40 bushels so that he may sell the mixture at 10 shillings per bushel?

Ans. 30 bushels at 9 shillings, and 10 at 13 shillings.

21. What number is that, the treble of which, increased by 12, shall as much exceed 54 as that treble is less than 144? Ans. 31.

22. A asked B how much money he had. He replied, if I had 5 times the sum I now possess, I could lend you $60, and then $\frac{1}{5}$ of the remainder would be equal to $\frac{1}{2}$ the dollars I now have. Required the sum which B had.

Ans. $24.

23. A, B, and C found a purse of money, and it was mutually agreed that A should receive $15 less than one half, that B should have $13 more than one quarter, and that C should have the remainder, which was $27. How many dollars did the purse contain? Ans. $100.

24. Two persons, A and B, 120 miles apart, set out at the same time to meet each other. A goes 3 miles an hour, and B 5 miles. What distance will each have traveled when they meet? Ans. A, 45 miles; B, 75 miles.

25. The first digit of a certain number exceeds the second by 4, and when the number is divided by the sum of the digits, the quotient is 7. What is the number?

Ans. 84.

26. A person bought a certain number of acres of land for $180. After reserving two of them, he sold the remainder for $180. Now he found that he had gained on the cost price of each acre one third more per cent. than that cost price. How many acres did he buy? Ans. 12.

27. A prize of $1000 is to be divided between A and B, so that their shares may be in the ratio of 7 to 8. Required the share of each.

Ans. A's share, $466\frac{2}{3}$, and B's, $533\frac{1}{3}$.

28. A gentleman let a certain sum of money for 3 years, at 5 per cent. compound interest; that is, at the end of each year there was added $\frac{1}{20}$ to the sum due. At the close of the third year there was due him $2315.25. Required the sum let. Ans. $2000.

29. Bought a picture at a certain price, and paid the same price for a frame; if the frame had cost $1 less, and the picture $0.75 more, the price of the frame would have been only half that of the picture. Required the cost of the picture. Ans. $2.75.

30. A and B can do a piece of work together in 7 days, which A alone could do in 10 days. In what time could B alone do it? Ans. $23\frac{1}{3}$ days.

31. A gentleman gave in charity $46; a part in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. Now, a man and a woman had between them $8. What was given to the men, and what to the women?

Ans. The men received $25, and the women $21.

32. A man has two farms, and his stock is worth $183. Now, the stock and his first farm are worth once and two sevenths the value of the second farm; and the stock and the second farm are worth once and five eighths the value of the first farm. What is the value of each farm?

Ans. First farm, $384; second farm, $441.

33. At 6 o'clock the hands of a watch are opposite the one to the other. When will they next be in like position?

Ans. At $5\frac{5}{11}$ minutes after 7 o'clock.

34. A vessel can be emptied by three taps; by the first alone it could be emptied in 80 minutes, by the second in 200 minutes, and by the third in 5 hours. In what time will it be emptied if all the taps be opened?

Ans. 48 minutes.

35. A fox is pursued by a greyhound, and is 60 of her own leaps before him. The fox makes 9 leaps while the

greyhound makes but 6; but the latter in 3 leaps goes as far as the former in 7. How many leaps does each make before the greyhound catches the fox?

Ans. The greyhound, 72 leaps; the fox, 108.

36. It is found that 7 men and 3 boys pulling together produce just the same effect as 9 men pulling *against* 9 boys. What is the relative strength of one of the men to one of the boys? Ans. 6 to 1.

37. It is required to divide the number 43 into two such parts that one of them shall be 3 times as much above 20 as the other wants of 17. Required the numbers. Ans. 29 and 14.

38. Two horses ran over a mile course; the winner completed the distance in 2 minutes 54 seconds, winning by 2 seconds. How many yards start might have been allowed to the other without risk of losing, supposing the same rates be kept? Ans. 20.

39. Gold is $19\frac{1}{4}$ times as heavy as water, and silver $10\frac{1}{2}$ times. A mixed mass weighs 4160 ounces, and displaces 250 ounces of water. What proportions of gold and silver does it contain?

Ans. Gold, 3377 ounces; silver, 783 ounces.

40. A alone could perform a piece of work in 12 hours; A and C together could do it in 5 hours; and C's work is $\frac{2}{3}$ of B's. Now, the work has to be completed by noon. A begins work at 5 o'clock in the morning; at what hour can he be relieved by B and C, and the work be just finished in time? Ans. At 10 o'clock.

41. A merchant possesses $5120, but at the beginning of each year he sets aside a fixed sum for family expenses. His business increases his capital employed therein annually at the rate of 25 per cent. At the end of four years he finds that his capital is reduced to $3275. What are his annual expenses? Ans. $1280.

SIMPLE EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

161. INDEPENDENT EQUATIONS are such as cannot be made to assume the same form.

If they relate to the same problem, they must, therefore, express essentially different conditions of that problem.

162. When the conditions of a problem require the unknown quantities to be denoted by different letters, in order to determine their values, as many independent equations are necessary as there are unknown quantities.

For, if we have an equation containing two unknown quantities, x and y, as

$$x - y = 1,$$

transposing y, we have

$$x = y + 1. \qquad (1)$$

But the value of y is not known; consequently, from this equation alone the value of x cannot be determined.

If, however, we have a second equation, as

$$x + y = 7,$$

or,
$$x = 7 - y, \qquad (2)$$

in which the value of x and y are the same as in the first, the second members of (1) and (2) being equal to the same quantity, x, and consequently equal to each other (Art. 46, Ax. 7), give

$$y + 1 = 7 - y;$$

or,
$$2y = 6.$$

Whence,
$$y = 3.$$

Substituting 3, the value of y, for y in either equation (1) or equation (2), we obtain 4 as the value of x; and

the values obtained for the two unknown quantities satisfy the two equations.

163. SIMULTANEOUS EQUATIONS are those in which the unknown quantities are satisfied by the same values.

Two unknown quantities, denoted by different letters, require for their determination, as has been shown (Art. 162), at least *two* independent, simultaneous equations.

ELIMINATION.

164. ELIMINATION is the process of combining simultaneous equations, in such a manner as to cause one or more of their unknown quantities to disappear.

There are four principal methods of elimination; — by *Comparison*, by *Substitution*, by *Addition or Subtraction*, and by *Undetermined Multipliers*.

CASE I.

165. Elimination by comparison.

1. Given $5x - 3y = 9$, and $2x + 5y = 16$, to find the values of x and y.

OPERATION.

$$5x - 3y = 9 \quad (1)$$
$$2x + 5y = 16 \quad (2)$$

From (1), by transposition and division, $x = \frac{9 + 3y}{5}$ (3)

From (2), by transposition, etc. $x = \frac{16 - 5y}{2}$ (4)

Combining (3) and (4), by Ax. 7, $\frac{9 + 3y}{5} = \frac{16 - 5y}{2}$ (5)

Clearing of fractions, $18 + 6y = 80 - 25y$

Or, $31y = 62$

Whence, $y = 2$

Substituting the value of y in (3), $x = 3$

Hence the

RULE,

Find the value of the same unknown quantity, in terms of the other, from each of the equations; and form a new equation by placing one of these two values equal to the other.

It is usually most convenient to reduce each equation, at first, to its simplest form.

CASE II.

166. Elimination by substitution.

2. Given $\frac{x}{2} - y = 1$, and $x - \frac{y}{2} = 8$, to find the values of x and y.

OPERATION.

$$\frac{x}{2} - y = 1 \qquad (1)$$

$$x - \frac{y}{2} = 8 \qquad (2)$$

Clearing (1) of fractions, $x - 2y = 2$

Or, $x = 2 + 2y \qquad (3)$

Substituting the value of x in (2), $2 + 2y - \frac{y}{2} = 8$

Clearing of fractions, $4 + 4y - y = 16$

Or, $3y = 12$

Whence, $y = 4$

Substituting the value of y in (3), $x = 10$

Hence the

RULE.

Find the value of one of the unknown quantities, in terms of the other, from either of the equations; and substitute this for the same quantity in the other equation.

The rule applies very advantageously when either of the unknown quantities has 1 for a coefficient.

CASE III.

167. Elimination by addition or subtraction.

3. Given $6x + 4y = 32$, and $4x - 2y = 12$, to find the values of x and y.

OPERATION.

$$6x + 4y = 32 \quad (1)$$
$$4x - 2y = 12 \quad (2)$$

Dividing (1) by 2, $3x + 2y = 16$ (3)

Adding (2) and (3), $7x = 28$

Whence, $x = 4$

Substituting the value of x in (3), $12 + 2y = 16$

Or, $2y = 4$

Whence, $y = 2$

4. Given $6x + 4y = 56$, and $4x - 3y = 9$, to find the values of x and y.

OPERATION.

$$6x + 4y = 56 \quad (1)$$
$$4x - 3y = 9 \quad (2)$$

Multiplying (1) by 2, $12x + 8y = 112$ (3)

Multiplying (2) by 3, $12x - 9y = 27$ (4)

Subtracting (4) from (3), $17y = 85$

Or, $y = 5$

Substituting the value of y in (2), $4x - 15 = 9$

Or, $4x = 24$

Whence, $x = 6$

Hence the

RULE.

Multiply or divide one or both of the equations, if necessary, by such a number or quantity that one of the unknown quantities shall have the same coefficient in both. Then, if the signs of the terms having the same coefficients are alike, subtract one equation from the other; or, if unlike, add the two equations.

When the coefficients of the quantity to be eliminated are prime to each other, each equation must be multiplied by the coefficient found in the other. In general, such a multiplier may be used for each as will produce the least common multiple of the coefficients.

CASE IV.

168. Elimination by undetermined multipliers.

An UNDETERMINED MULTIPLIER is a factor, at first undetermined, but to which a convenient value is assigned in the course of the operation.

5. Given $4x + 2y = 20$, and $6x + 4y = 32$, to find the values of x and y.

OPERATION.

$$4x + 2y = 20 \quad (1)$$
$$6x + 4y = 32 \quad (2)$$

Multiplying (1) by m, $4mx + 2my = 20m$ (3)

Subtracting (2) from (3), and factoring with reference to x and y,

$$(4m - 6)x + (2m - 4)y = 20m - 32 \quad (4)$$

Assuming $2m - 4 = 0$, the second term of (4) disappears, and

$$(4m - 6)x = 20m - 32$$

Or,
$$x = \frac{20m - 32}{4m - 6}$$

Whence, since $2m - 4 = 0$, $m = 2$, $x = 4$

Substituting the value of x in (1), $y = 2$

Instead of subtracting (2) from (3), we might have added them, and obtained like results.

In (4), x might have been eliminated by assuming $4m - 6 = 0$; but the value of m would have been $\frac{3}{2}$.

Now, if we multiply the first of the given equations by 2, the first value of m, the coefficient of y in it will

become the same as in the second equation; and, if we divide the second of the given equations by $\frac{3}{2}$, the second value of m, the coefficient of x in it will become the same as in the first. Hence, this may be regarded as a generalization of the method by addition or subtraction.

RULE.

Multiply one of the given equations by the undetermined quantity m, *and add to, or subtract from, the result, the other given equation.*

In the resulting equation, factored with reference to the unknown quantities, place the coefficient of the quantity first to be eliminated equal to zero, and find the value of m. *Substitute this value in the equation containing the uneliminated unknown quantity and* m, *and reduce the equation.*

169. Solve by whichever method may be most advantageous, the following

EXAMPLES.

Find the values of the unknown quantities in each of the following equations.

6. Given $\left\{\begin{matrix} 3x+7y=33 \\ 2x+4y=20 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=4. \\ y=3. \end{matrix}\right.$

7. Given $\left\{\begin{matrix} 7x+2y=39 \\ 3x-4y=7 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=5. \\ y=2. \end{matrix}\right.$

8. Given $\left\{\begin{matrix} 6x-3y=27 \\ 4x-6y=-2 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=7. \\ y=5. \end{matrix}\right.$

9. Given $\left\{\begin{matrix} 7x+3y=62 \\ 5x-2y=36 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=8. \\ y=2. \end{matrix}\right.$

10. Given $\left\{\begin{matrix} 12x+8y=116 \\ 2x-y=3 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=5. \\ y=7. \end{matrix}\right.$

11. Given $\left\{\begin{matrix} 11x+3y=124 \\ 2x-6y=-56 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=8. \\ y=12. \end{matrix}\right.$

12. Given $\left\{\begin{matrix} 9x+4y=58 \\ 3x+2y=26 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=2. \\ y=10. \end{matrix}\right.$

13. Given $\left\{\begin{matrix} 6x+5y=112 \\ 8x-2y=80 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=12. \\ y=8. \end{matrix}\right.$

14. Given $\left\{\begin{matrix} 7x-2y=-6 \\ 2x+2y=24 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=2. \\ y=10. \end{matrix}\right.$

15. Given $\left\{\begin{matrix} 6x+11y=115 \\ 8x-22y=-30 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=10. \\ y=5. \end{matrix}\right.$

16. Given $\left\{\begin{matrix} 2x+3y=47 \\ 10x-12y=-62 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=7. \\ y=11. \end{matrix}\right.$

17. Given $\left\{\begin{matrix} \frac{x}{2}-\frac{y}{3}=0 \\ \frac{x}{4}+\frac{y}{6}=6 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=12. \\ y=18. \end{matrix}\right.$

18. Given $\left\{\begin{matrix} \frac{3x}{5}-y=11 \\ x+\frac{y}{5}=37 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=35. \\ y=10. \end{matrix}\right.$

19. Given $\left\{\begin{matrix} \frac{4x}{7}-\frac{2y}{3}=2 \\ x+7y=175 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=28. \\ y=21. \end{matrix}\right.$

20. Given $\left\{\begin{matrix} \frac{7x}{8}-\frac{y}{9}=19 \\ 3x+3y=126 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=24. \\ y=18. \end{matrix}\right.$

21. Given $\left\{\begin{matrix} 14x+\frac{5y}{6}=38 \\ x+12y=146 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=2. \\ y=12. \end{matrix}\right.$

22. Given $\left\{\begin{matrix} \frac{x}{7}-\frac{7y}{10}=-20 \\ \frac{x}{4}+3y=134 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=56. \\ y=40. \end{matrix}\right.$

23. Given $\left\{\begin{matrix} a\,x+b\,y=c \\ a'\,x+b'\,y=c' \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=\frac{b'\,c-b\,c'}{a\,b'-a'\,b}. \\ y=\frac{a\,c'-a'\,c}{a\,b'-a'\,b}. \end{matrix}\right.$

24. Given $\left\{\begin{matrix} \frac{x}{a}-\frac{y}{b}=m \\ \frac{x}{c}+\frac{y}{d}=n \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=\frac{a\,b\,c\,m+a\,c\,d\,n}{a\,d+b\,c}. \\ y=\frac{b\,c\,d\,n-a\,b\,d\,m}{a\,d+b\,c}. \end{matrix}\right.$

25. Given $\left\{\begin{matrix} \frac{x}{2}-12=\frac{y}{4}+8 \\ \frac{x+y}{5}+\frac{x}{3}-8=\frac{2\,y-x}{4}+27 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=60. \\ y=40. \end{matrix}\right.$

SIMPLE EQUATIONS

CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

170. The methods given for the solution of simultaneous equations containing two unknown quantities may be extended to those containing three or more unknown quantities.

1. Given $\left\{\begin{matrix} 10\,x-6\,y+4\,z=30 \\ 2\,x+\ y-\ z=\ 9 \\ 5\,x+4\,y-2\,z=28 \end{matrix}\right\}$ to find x, y, and z.

OPERATION.

$$10\,x-6\,y+4\,z=30 \quad (1)$$
$$2\,x+\ y-\ z=\ 9 \quad (2)$$
$$5\,x+4\,y-2\,z=28 \quad (3)$$

Dividing (1) by -2, $-5\,x+3\,y-2\,z=-15$ (4)

Multiplying (2) by 2, $4\,x+2\,y-2\,z=18$ (5)

Subtracting (4) from (3), $10\,x+y=43$ (6)

Subtracting (5) from (4), $-9\,x+y=-33$ (7)

Subtracting (7) from (6), $19\,x=76$

Or, $x=4$

Substituting the value of x in (6), $y=3$

Substituting the values of x and y in (2), $z=2$

2. Given $\left\{\begin{matrix} x+y+z=53 \\ x+2y+3z=107 \\ x+3y+4z=137 \end{matrix}\right\}$ to find x, y, and z.

OPERATION.

$$\begin{matrix} x+y+z=53 & (1) \\ x+2y+3z=107 & (2) \\ x+3y+4z=137 & (3) \end{matrix}$$

Subtracting (1) from (2), $y+2z=54$ (4)
Subtracting (2) from (3), $y+z=30$ (5)

Subtracting (5) from (4), $z=24$
Substituting the value of z in (5), $y=6$
Substituting the values of y and z in (1), $x=23$

From these examples we deduce the following

RULE.

Combine the given equations, so as to form a new set of equations, with a less number of unknown quantities; and continue the process with the new equations; and so on, until an equation is obtained containing but one unknown quantity.

Find the value of the unknown quantity in this equation. By substituting this value in either of the equations containing only two unknown quantities, find the value of a second unknown quantity. Then, by substituting these values in either of the equations which contain three unknown quantities, find the value of a third; and so on, till the values of all are found.

EXAMPLES.

3. Given $\left\{\begin{matrix} 3x-y-2z=0 \\ 6x+2y+3z=45 \\ 4x+3y-z=31 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=4. \\ y=6. \\ z=3. \end{matrix}\right.$

4. Given $\left\{\begin{matrix} 8x-9y-7z=-36 \\ 12x-y-3z=36 \\ 6x-2y-z=10 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=4. \\ y=6. \\ z=2. \end{matrix}\right.$

5. Given $\left\{\begin{array}{l} 7x+4y-\ z=\ 78 \\ 4x-5y-3z=-21 \\ x-3y-4z=-37 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=8. \\ y=7. \\ z=6. \end{array}\right.$

6. Given $\left\{\begin{array}{l} u+x+y=6 \\ u+x+z=9 \\ u+y+z=8 \\ x+y+z=7 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} u=3. \\ x=2. \\ y=1. \\ z=4. \end{array}\right.$

The solution may here be abridged, by the artifice of assuming the sum of the four unknown quantities to equal an auxiliary quantity, s. Thus,

Let	$u+x+y+z=s$	
Then	$s-z=6$	(1)
	$s-y=9$	(2)
	$s-x=8$	(3)
	$s-u=7$	(4)
By addition,	$4s-s=30$	(5)
Whence,	$s=10$	(6)

Substituting the value of s in (4), (3), (2), and (1), we have $u=3$, $x=2$, $y=1$, and $z=4$.

7. Given $\left\{\begin{array}{l} x+y=30 \\ x+z=25 \\ y+z=15 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=20. \\ y=10. \\ z=\ 5. \end{array}\right.$

8. Given $\left\{\begin{array}{l} 8x-4y=24-\ z \\ 6x+\ y=\ z+84 \\ x+80=3y+4z \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=12. \\ y=20. \\ z=\ 8. \end{array}\right.$

9. Given $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}-\frac{z}{4}=23 \\ \frac{x}{3}-\frac{y}{4}+\frac{z}{2}=12 \\ \frac{x}{4}+\frac{y}{2}-\frac{z}{3}=17 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=36. \\ y=24. \\ z=12. \end{array}\right.$

10. Given $\left\{\begin{array}{l} 3u+x+2y-z=22 \\ \quad 4x-y+3z=35 \\ 4u+3x-2y\quad\;=19 \\ 2u\quad\;+4y+2z=46 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} u=4. \\ x=5. \\ y=6. \\ z=7. \end{array}\right.$

11. Given $\left\{\begin{array}{l} a=y+z \\ b=x+z \\ c=x+y \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=\frac{1}{2}\,(b+c-a). \\ y=\frac{1}{2}\,(a+c-b). \\ z=\frac{1}{2}\,(a+b-c). \end{array}\right.$

12. Given $\left\{\begin{array}{l} 4x-4y=a+4z \\ 6y-2x=a+2z \\ 7z-\;\;y=a+\;\;x \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=\frac{13}{8}\,a. \\ y=\frac{7}{8}\,a. \\ z=\frac{1}{2}\,a. \end{array}\right.$

13. Given $\left\{\begin{array}{l} 2x+2y+\;\;z=35-2u \\ \quad\;\;y+3z=31 \\ \quad 4x+\;\;z=32 \\ \quad \frac{1}{3}x+3y=23 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} u=\frac{1}{2}. \\ x=6. \\ y=7. \\ z=8. \end{array}\right.$

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

1. If I were to enlarge my field by making it 5 rods longer and 4 rods wider, its area would be increased 240 square rods; but if I were to make its length 4 rods less, and its width 5 rods less, its area would be diminished 210 square rods. Required the present length, width, and area.

SOLUTION.

Let x = present length, in rods,

and y = present width, in rods.

Then xy = present area, in sq. rods.

By the conditions, $(x+5)(y+4) = xy + 240$ (1)

and $(x-4)(y-5) = xy - 210$ (2)

From (1), $xy + 5y + 4x + 20 = xy + 240$ (3)

From (2), $xy - 4y - 5x + 20 = xy - 210$ (4)

Transposing (3), $5y + 4x = 220$ (5)

Transposing (4), $4y + 5x = 230$ (6)

Multiplying (5) by 4, $20y + 16x = 880$ (7)

Multiplying (6) by 5, $20y + 25x = 1150$ (8)

Subtracting (7) from (8), $9x = 270$

Whence, $x = 30$, present length in rds.

Substituting value of x in (6), $y = 20$, present width in rds.,

and $xy = 600$, present area in rds.

2. A boatman can row down stream, a distance of 20 miles, and back again, in 10 hours, the current being uniform all the time; and he finds that he can row 2 miles against the current in the same time that he rows 3 miles with it. Required the time in going and in returning.

SOLUTION.

Let x = rate per hour rowing without current,

and y = rate per hour of the current.

Then $x + y$ = rate down per hour,

and $x - y$ = rate up per hour.

Then $\frac{20}{x+y}$ = number of hours in going,

and $\frac{20}{x-y}$ = number of hours in returning.

By the conditions, $\frac{20}{x+y} + \frac{20}{x-y} = 10$ (1)

and $\frac{2}{x-y} = \frac{3}{x+y}$ (2)

From (1), $\frac{2x}{x^2-y^2} = \frac{1}{2}$ (3)

From (2), $x = 5y$ (4)

Substituting value of x in (3), $\frac{10y}{25y^2-y^2} = \frac{1}{2}$ (5)

Or, $20y = 24y^2$ (6)

Whence, $y = \frac{5}{6}$ (7)

From (4) and (7), $\frac{20}{x+y} = \frac{20}{6y} = 4$, no. hrs. going;

Also, $\frac{20}{x-y} = \frac{20}{4y} = 6$, no. of hours in returning.

3. A and B can perform a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. In how many days can each of them alone perform it?

SOLUTION.

Let $x =$ no. days in which A alone can perform it,

$y =$ no. days in which B alone can perform it,

and $z =$ no. days in which C alone can perform it.

Then $\frac{1}{x} = x^{-1} =$ the part which A can perform in one day,

$\frac{1}{y} = y^{-1} =$ the part which B can perform in one day,

and $\frac{1}{z} = z^{-1} =$ the part which C can perform in one day.

Also, $\frac{1}{6} =$ the part A and B can perform in one day,

$\frac{1}{8} =$ the part A and C can perform in one day,

and $\frac{1}{12} =$ the part B and C can perform in one day.

By the conditions, $x^{-1}+y^{-1}=\frac{1}{6}$ (1)

$x^{-1}+z^{-1}=\frac{1}{8}$ (2)

and $y^{-1}+z^{-1}=\frac{1}{12}$ (3)

Adding (1), (2), and (3), $2x^{-1}+2y^{-1}+2z^{-1}=\frac{3}{8}$. (4)

Dividing (4) by 2, $x^{-1}+y^{-1}+z^{-1}=\frac{3}{16}$ (5)

Subtracting (3) from (5), $x^{-1}=\frac{5}{48}$ (6)

Subtracting (2) from (5), $y^{-1}=\frac{1}{16}$ (7)

Subtracting (1) from (5), $z^{-1}=\frac{1}{48}$ (8)

From (6), $x=9\frac{3}{5}$, no. of days in which A can perform it.

From (7), $y=16$, no. of days in which B can perform it.

From (8), $z=48$, no. of days in which C can perform it.

4. A says to B, if $\frac{1}{5}$ of my age were added to $\frac{2}{3}$ of yours, the sum would be $19\frac{1}{3}$ years. But, says B, if $\frac{2}{5}$ of mine were subtracted from $\frac{7}{8}$ of yours, the remainder would be $18\frac{1}{4}$ years. Required their ages.

Ans. A's age, 30 years; B's, 20 years.

5. If 1 be added to the numerator of a certain fraction, its value is $\frac{1}{3}$; but if 1 be added to its denominator, its value is $\frac{1}{4}$. What is the fraction? Ans. $\frac{4}{15}$.

6. A farmer had 89 oxen and cows; but, having sold 4 oxen and 20 cows, found he then had 7 more oxen than cows. Required the number of each at first.

Ans. 40 oxen; 49 cows.

7. A says to B, if 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be \$990. B replied, if 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be \$510. Required the property of each.

Ans. A's, \$140; B's, \$70.

8. If $\frac{1}{7}$ of A's age were subtracted from B's age, and 5 years added to the remainder, the sum would be 6 years; and if 4 years were added to $\frac{1}{5}$ of B's age, it would be equal to $\frac{1}{14}$ of A's age. Required their ages.

Ans. A's, 98 years; B's, 15 years.

9. It is required to divide 50 into two such parts that $\frac{3}{5}$ of the larger shall be equal to $\frac{2}{3}$ of the smaller.

Ans. 32 and 18.

10. A gentleman, at the time of his marriage, found that his wife's age was to his as 3 to 4; but, after they had been married 12 years, her age was to his as 5 to 6. Required their ages at the time of their marriage.

Ans. The man's age, 24; his wife's, 18 years.

11. A farmer hired a laborer for ten days, and agreed to pay him \$12 for every day he labored, and he was to forfeit \$8 for every day he was absent. He received at the end of his time \$40. How many days did he labor, and how many days was he absent?

Ans. He labored 6 days; was absent, 4.

12. A gentleman bought a horse and chaise for \$208, and $\frac{4}{7}$ of the cost of the chaise was equal to $\frac{2}{3}$ the price of the horse. What was the price of each?

Ans. Chaise, \$112; horse, \$96.

13. A and B engaged in trade, A with \$240, and B with \$96. A lost twice as much as B; and, upon settling their accounts, it appeared that A had three times as much remaining as B. How much did each lose?

Ans. A lost \$96; B lost \$48.

14. Two men, A and B, agreed to dig a well in 10 days, but, having labored together 4 days, B agreed to finish the job, which he did in 16 days. How long would it have required A to dig the whole well? Ans. 16 days.

15. A merchant has two kinds of grain, one at 60 cents per bushel, and the other at 90 cents per bushel, of which he wishes to make a mixture of 40 bushels that may be worth 80 cents per bushel. How many bushels of each must he use?

Ans. $13\frac{1}{3}$ bushels at 60 cents; $26\frac{2}{3}$ at 90 cents.

16. A farmer has a large box, filled with wheat and rye; seven times the bushels of wheat are 3 bushels more than four times the bushels of rye; and the quantity of wheat is to the quantity of rye as 3 to 5. Required the bushels of wheat and the bushels of rye.

Ans. Wheat, 9 bushels; rye, 15 bushels.

17. My income and assessed taxes together amount to $50. But if the income tax be increased 50 per cent., and assessed tax diminished 25 per cent., the taxes will together amount to $52.50. Required the amount of each tax. Ans. Income tax, $20; assessed tax, $30.

18. A and B entered into partnership, and gained $200. Now, 6 times A's accumulated stock (capital and profit) was equal to 5 times B's original stock; and 6 times B's profit exceeded A's original stock by $200. Required the original stock of each.

Ans. A's stock, $500; B's stock, $700.

19. A boy at a fair spent his money for oranges. If he had got five more for his money, they would have averaged a half-cent each less; and if three less, a half-cent each more. How many cents did he spend, and how many oranges did he get?

Ans. 30 cents; 15 oranges.

20. A merchant has three kinds of sugar. He can sell 3 lbs. of the first quality, 4 lbs. of the second quality, and 2 lbs. of the third quality, for 60 cents; or, he can sell 4 lbs. of the first quality, 1 lb. of the second quality, and 5 lbs. of the third quality, for 59 cents; or, he can sell 1 lb. of the first quality, 10 lbs. of the second quality, and 3 lbs. of the third quality, for 90 cents. Required the price of each quality.

Ans. First quality, 8 cents per lb.; second, 7 cents; third, 4 cents.

21. A gentleman's two horses, with their harness, cost him $120. The value of the poorer horse, with the har-

ness, was double that of the better horse; and the value of the better horse, with the harness, was triple that of the poorer horse. What was the value of each?

Ans. Harness, $50; better horse, $40; poorer, $30.

22. Find three numbers, so that the first with half the other two, the second with one third of the other two, and the third with one fourth of the other two, shall each be equal to 34. Ans. 10, 22, and 26.

23. Find a number of three places, of which the digits have equal differences in their order; and, if the number be divided by half the sum of the digits, the quotient will be 41; and, if 396 be added to the number, the digits will be inverted. Ans. 246.

24. There are 4 men, A, B, C, and D, the value of whose estates is $14,000; twice A's, three times B's, half of C's, and one fifth of D's, is $16,000; A's, twice B's, twice C's, and two fifths of D's, is $18,000; and half of A's, with one third of B's, one fourth of C's, and one fifth of D's, is $4000. Required the property of each.

Ans. A's, $2000; B's, $3000; C's, $4000; D's, $5000.

25. A and B driving their turkeys to market, A says to B, give me 5 of your turkeys, and I shall have as many as you. B replies, but give me 15 of yours, and then yours will be $\frac{3}{7}$ of mine. What number of turkeys had each? Ans. A, 45 turkeys; B, 55 turkeys.

26. A person possesses a capital of $30,000, on which he gains a certain rate of interest; but he owes $20,000, for which he pays interest at another rate. The interest which he receives is greater than that which he pays by $800. A second person has $35,000, on which he gains the second rate of interest; but he owes $24,000, for which he pays the first rate of interest. The sum which he receives is greater than that which he pays by $310. What are the two rates of interest?

Ans. First rate, 6 per cent.; second rate, 5 per cent.

27. A says to B and C, give me half of your money, and I shall have $55. B replies, if you two will give me one third of yours, I shall have $50. But C says to A and B, if I had one fifth of your money, I should have $50. Required the sum that each possessed.

Ans. A, $20; B, $30; C, $40.

28. A gentleman left a sum of money to be divided among his four sons, so that the share of the oldest was $\frac{1}{2}$ of the sum of the shares of the other three, the share of the second $\frac{1}{3}$ of the sum of the other three, and the share of the third $\frac{1}{4}$ of the sum of the other three; and it was found that the share of the oldest exceeded that of the youngest by $14. What was the whole sum, and what was the share of each person?

Ans. Whole sum, $120; oldest son's share, $40; second son's, $30; third son's, $24; youngest son's, $26.

GENERAL SOLUTION OF PROBLEMS.

171. In the GENERAL SOLUTION of a problem, all the quantities are represented by letters, or the symbols of general values.

The *unknown* thus found in terms of the *known* quantities is a general expression, or *formula*, which can be used for the solution of any similar problem.

172. A problem is said to be *generalized* when letters are used to represent its known quantities.

The algebraic solution of general problems discloses many interesting truths and useful practical rules, as may be seen from the consideration of the following, among other

General Problems.

1. The sum of two numbers is a, and their difference is b; what are the two numbers?

SOLUTION.

Let $x =$ the greater number,
and $y =$ the less number.

By the conditions, $x + y = a$ (1)
Also, $x - y = b$ (2)

Adding (1) and (2), $2x = a + b$ (3)
Subtracting (2) from (1), $2y = a - b$ (4)

From (3), $x = \frac{a+b}{2}$, the greater number.

From (4), $y = \frac{a-b}{2}$, the less number.

Hence, since a and b may have any value whatever, the values of x and y are general, and may be expressed as rules for the numerical calculations in any like case; thus,

To find two numbers, when their sum and difference are given, — *Add the sum and difference, and divide by 2, for the greater of the two numbers; subtract the difference from the sum, and divide by 2, for the less number.*

2. A can do a piece of work in a days, which it requires b days for B to perform. In how many days can it be done if A and B work together?

SOLUTION.

Let $x =$ the number of days required, and $1 =$ the entire work; then, in 1 day A can do $\frac{1}{a}$ of the work, and B $\frac{1}{b}$; therefore, in x days, they can do $\frac{x}{a}$ and $\frac{x}{b}$ of the work. Hence,

By the conditions, $\frac{x}{b} + \frac{x}{a} = 1$

Clearing of fractions, $ax + bx = ab$

Or, $(a + b)\, x = ab$

Whence, $x = \frac{ab}{a+b}$, no. days required.

Hence,

To find the time required for two agencies conjointly to accomplish a certain result, when the times are given in which each separately can accomplish the same, — *Divide the product of the given times by their sum.*

3. A cistern can be filled by three pipes; by the first in a hours, by the second in b hours, and by the third in c hours. In what time can it be filled by all the pipes running together?

Ans. $\frac{abc}{ab + ac + bc}$ hours.

Here it will be seen that, when three agencies are employed, the required time *is the product of the given times, divided by the sum of their products, taken two and two.*

4. In the last example, what will be the time required, if $a = 2$, $b = 5$, and $c = 10$? Ans. $1\frac{1}{4}$ hours.

5. Three men, A, B, and C, enter into partnership for a certain time. Of the capital stock A furnishes m dollars; B, n dollars; and C, p dollars. They gain a dollars. What is each man's share of the gain?

SOLUTION.

Let $x =$ A's share.

Then $\frac{nx}{m} =$ B's share,

and $\frac{px}{m} =$ C's share.

By the conditions, $x + \frac{nx}{m} + \frac{px}{m} = a$

Clearing of fractions, $mx + nx + px = ma$

Or, $(m + n + p)\, x = ma$

Whence, $x = \frac{ma}{m+n+p}$, A's share.

Then $\frac{nx}{m} = \frac{na}{m+n+p}$, B's share.

Also, $\frac{px}{m} = \frac{pa}{m+n+p}$, C's share.

Hence,

To find each man's gain, when each man's stock and the whole gain are given, — *Multiply the whole gain by each man's stock, and divide the product by the whole stock.*

6. In the last example, if $m = \$300$, $n = \$500$, $p = \$800$, and $a = \$320$, what is each man's share of the gain?

Ans. A's share, \$60; B's share, \$100; C's share, \$160.

7. Divide the number a into two parts which shall have to each other the ratio of m to n.

Ans. $\frac{ma}{m+n}$ and $\frac{na}{m+n}$.

8. A courier left this place n days ago, and goes a miles each day. He is pursued by another, starting to-day and going b miles daily. How many days will the second require to overtake the first?

Ans. $\frac{na}{b-a}$ days.

9. Required what principal at interest at r per cent. will amount to the sum a, in t years.

Ans. $\frac{100a}{100+rt}$.

10. A gentleman distributing some money among beggars, found that in order to give them a cents each, he should want b cents more; he therefore gave them c cents each, and he had d cents left. Required the number of beggars.

Ans. $\frac{b+d}{a-c}$.

11. Required the number of years in which p dollars, at r per cent. interest, will amount to a dollars.

Ans. $\frac{100\,(a-p)}{rp}$.

12. A banker has two kinds of money; it takes a pieces of the first to make a dollar, and b pieces of the second to make the same sum. If he is offered a dollar for c pieces, how many of each kind must he give?

Ans. First kind, $\frac{a\,(c-b)}{a-b}$; second kind, $\frac{b\,(a-c)}{a-b}$.

13. In the last example, if $a=10$, $b=20$, and $c=15$, how many of each kind must he give?

Ans. First kind, 5; second kind, 10.

14. A mixture is made of a pounds of coffee at m cents a pound, b pounds at n cents, and c pounds at p cents. Required the cost per pound of the mixture.

Ans. $\frac{ma+nb+pc}{a+b+c}$.

15. A, B, and C hire a pasture together for a dollars. A puts in m horses for t months, B puts in n horses for t' months, and C puts in p horses for t'' months. What part of the expense should each pay?

Ans. A, $\$\frac{mta}{mt+nt'+pt''}$; B, $\$\frac{nt'a}{mt+nt'+pt''}$;
C, $\$\frac{pt''a}{mt+nt'+pt''}$.

DISCUSSION OF PROBLEMS

LEADING TO SIMPLE EQUATIONS.

173. The Discussion of a problem, or of an equation, is the process of attributing any reasonable values and relations to the arbitrary quantities which enter the equation, and interpreting the results.

174. An Arbitrary Quantity is one to which any reasonable value may be given at pleasure.

175. A DETERMINATE PROBLEM is one in which the given conditions furnish the means of finding the required quantities.

A determinate problem leads to as many independent equations as there are required quantities (Art. 162).

176. An INDETERMINATE PROBLEM is one in which there are fewer imposed conditions than there are required quantities, and, consequently, an insufficient number of independent equations to determine definitely the values of the required quantities.

177. An IMPOSSIBLE PROBLEM is one in which the conditions are incompatible or contradictory, and consequently cannot be fulfilled.

178. A determinate problem, leading to a simple equation involving only one unknown quantity, can be satisfied by but *one* value of that unknown quantity.

For, after transposing and uniting, if the coefficient of x is represented by a, and the second member by b, the equation will take the general form,

$$ax = b. \qquad (1)$$

Suppose now, if possible, this equation has two roots, r and r', then, by substituting successively these values in (1),

$$ar = b, \qquad (2)$$

$$ar' = b. \qquad (3)$$

Therefore, by subtracting (3) from (2), we have

$$ar - ar' = 0,$$

Or,
$$a(r - r') = 0.$$

But the last equation is impossible, since by supposition $r - r'$ is not zero, and a is not zero. That is,

A determinate simple equation can have but one root.

179. An indeterminate problem, or one leading to a less number of independent equations than it has unknown quantities, may be satisfied by any number of values.

For example, suppose a problem involving three unknown quantities leads only to two equations, which, on combining, give

$$x - z = 10,$$

or,

$$x = 10 + z.$$

Now, if we make

$$z = 1, \text{ then } x = 11;$$
$$z = 2, \text{ then } x = 12;$$
$$z = 3, \text{ then } x = 13.$$

Thus, we may find sets of values without limit that will satisfy the equation. Hence,

An indeterminate equation may have an infinite number of solutions.

180. When a problem leads to more independent equations than it has unknown quantities to be determined, it is impossible.

For, suppose we have a problem furnishing three independent equations, as,

$$x = y + 1 \qquad (1)$$
$$y = 7 - x \qquad (2)$$
$$xy = 16 \qquad (3)$$

From (1) and (2), we have

$$x = 4, \text{ and } y = 3.$$

But (3) requires their product to be 16, which can not be satisfied; hence the problem is impossible.

If, however, the third equation had not been independent, but derived from the other two; as,

$$xy = 12;$$

then the problem would have been possible, but the last equation, not being required for the solution (Art. 162), would have been *redundant.*

INTERPRETATION OF NEGATIVE RESULTS.

181. In a NEGATIVE RESULT, or a result preceded by the sign —, the negative sign is regarded as a symbol of *interpretation.*

Its significance when thus used it is now proposed to investigate.

1. Let it be required to find what number must be added to the number a, that the sum may be b.

Let	$x =$ the required number.
Then,	$a + x = b.$
Whence,	$x = b - a.$

Here, the value of x corresponds to any assigned values of a and b. Thus, for example,

Let	$a = 12$ and $b = 25.$
Then,	$x = 25 - 12 = 13,$

which satisfies the conditions of the problem; for if 13 be added to 12, or a, the sum will be 25, or b.

But suppose	$a = 30$ and $b = 24.$
Then	$x = 24 - 30 = -6,$

which indicates that, under the latter hypothesis, the problem is impossible in an *arithmetical sense,* though it is possible in the *algebraic sense* of the words "number," "added," and "sum."

The negative result, -6, points out, therefore, either *an error* or *an impossibility.*

But, taking the value of x with a contrary sign, we see that it will satisfy the enunciation of the problem, in an arithmetical sense, when modified so as to read:

What number must be taken from 30, that *the difference* may be 24?

2. Let it be required to find the epoch at which A's age is twice as great as B's, A's age at present being 35 years, and B's 20 years.

Let us suppose the required epoch to be *after* the present date.

Then $x =$ the number of years *after* the present date,
and $35 + x = 2\,(20 + x)$;
whence, $x = -5$,
a negative result.

On recurring to the problem, we find it is so worded as to admit also of the supposition that the epoch is *before* the present date, and taking the value of x obtained, with the contrary sign, we find it will satisfy that enunciation.

Hence, a negative result here indicates that a wrong choice was made of two possible suppositions which the problem allowed.

From the discussion of these problems we may infer:—

1. *That negative results indicate either an erroneous enunciation of a problem, or a wrong supposition respecting the* QUALITY *of some quantity belonging to it.*

2. *That we may form, when attainable, a possible problem analogous to that which involved the impossibility, or correct the wrong supposition, by attributing to the unknown quantity in the equation a* QUALITY DIRECTLY OPPOSITE *to that which had been attributed to it.*

In general, it is not necessary to form a *new* equation, but simply to change in the *old* one the sign of each quantity which is to have its quality changed.

Interpret the negative results obtained, and modify the enunciation accordingly, for the following

PROBLEMS.

3. If the length of a field be 10 rods, and the breadth 8 rods, what quantity must be added to its breadth that the contents be 60 square rods? Ans. — 2 rods.

4. If 1 be added to the numerator of a certain fraction, its value becomes $\frac{4}{5}$; but if 1 be added to the denominator, it becomes $\frac{5}{8}$. What is the fraction? Ans. $\frac{-5}{-9}$.

5. The sum of two numbers is 90, and their difference is 120; what are the numbers? Ans. 105 and — 15.

6. A is 50 years old, and B 40; required the time when A will be twice as old as B.

Ans. — 30 years.

7. A and B were in partnership, and A had 3 times as much capital in the firm as B. When A had gained $2000, and B $750, A had twice as much capital as B. What was the capital of each at first?

Ans. A was in *debt* $1500, and B $500.

8. A man worked 14 days, his son being with him 6 days, and received $39, besides the subsistence of himself and son while at work. At another time he worked 10 days, and had his son with him 4 days, and received $28. What were the daily wages of each?

Ans. The father's wages, $3; the son's, — 50 cts.

That is, the father earned $3 a day, and was at the expense of 50 cents a day for his son's subsistence.

9. A worked 10 days, B 4 days, and C 3 days, and their wages amounted to $23; at another time, A worked 9 days, B 8 days, and C 6 days, and their wages amounted to $24; a third time, A worked 7 days, B 6 days, and C 4 days, and their wages amounted to $18. What were the daily wages of each?

Ans. A's, $2.00; B's, 0; and C's, $1.00.

ZERO AND INFINITY.

182. ZERO, or the symbol 0, not only denotes absence of value, or nothing, but may, in Algebra, stand for a quantity less than any assignable value.

INFINITY, or the symbol ∞, denotes a quantity greater than any assignable value.

183. Hence, zero and infinity constitute the limits of all absolute or finite values; and, in comparison with infinities, finite values may be considered as all equal.

INTERPRETATION OF $\frac{A}{0}$, $\frac{A}{\infty}$, $\frac{0}{A}$, AND $\frac{0}{0}$.

184. In order to explain the meaning of these symbols, let us take the fraction $\frac{a}{b}$.

1. Suppose the numerator, a, to remain constant, while the denominator, b, continually decreases. Then, since the value of a fraction depends upon the relative value of its terms (Art. 119), the fraction must increase as the denominator decreases; consequently, when b decreases below any determinate limits, the value of the fraction must exceed any determinate or assignable quantity. Hence, representing any finite quantity by A, we have

$$\frac{A}{0} = \infty.$$

That is,

If a finite quantity is divided by zero, the quotient is infinity.

2. Suppose the numerator, a, to remain constant, while the denominator, b, constantly increases. Then, the value of the fraction must decrease as the denominator increases; consequently, when b increases beyond any determinate limits, the value of the fraction must be less than any determinate or assignable quantity. Hence we have

$$\frac{A}{\infty} = 0.$$

That is,

If a finite quantity is divided by infinity, the quotient is zero.

3. Suppose, now, the denominator, b, to remain constant, while the numerator, a, constantly decreases. Then the value of the fraction must decrease as the numerator decreases; consequently, when a decreases below any determinate limits, the value of the fraction must be less than any determinate or assignable quantity. Hence we have

$$\frac{0}{A} = 0.$$

That is,

If zero is divided by a finite quantity, the quotient is zero.

4. Suppose, next, a and b both to decrease, at the same time and in the same ratio. Then, the value of the fraction will not be changed; but when a and b decrease below any determinate limits, the terms of the fractions each become zero, and the fraction itself becomes $\frac{0}{0}$. As $\frac{a}{b}$ may have any value, $\frac{0}{0}$ will represent any finite quantity. Hence,

If zero is divided by zero, the quotient may be any finite quantity.

It is not true, however, that $\frac{0}{0}$ is always a symbol of an indeterminate finite quantity.

For, if $\frac{0}{0}$ is the result of an expression whose numerator contained more zero factors than its denominator, its value is 0; or if its denominator contained more zero factors than its numerator, its value is ∞. Thus, when $a = b$,

$$\frac{(a-b)^2}{a^2-b^2} = \frac{0}{0} = \frac{a-b}{a+b} = \frac{0}{2a} = 0,$$

$$\frac{a^3-b^3}{(a-b)^2} = \frac{0}{0} = \frac{a^2+ab+b^2}{(a-b)^2} = \frac{3a^2}{0} = \infty.$$

Also, $\frac{0}{0}$, by canceling a common factor in the terms of the fraction from which it originates, is sometimes found to have a definite finite value. Thus, when $a = b$,

$$\frac{a^3 + a^2 b - a b^2 - b^3}{a^2 - b^2} = \frac{0}{0} = \frac{(a+b)(a^2 - b^2)}{a^2 - b^2} = a + b = 2a.$$

The expressions $0 \times \infty$ and $\frac{\infty}{\infty}$ may also have the same four values that have just been attributed to $\frac{0}{0}$.

NOTE. — A proper understanding of the theory of indetermination, and of the relations of 0 to finite quantities, will lead to the detection of the fallacy in some *apparently* remarkable results. For instance,

Let	$a = b$	(1)
Multiplying both members by a, and subtracting b^2 from both,	$a^2 - b^2 = ab - b^2$	(2)
Factoring,	$(a+b)(a-b) = b(a-b)$	(3)
Dividing by $a-b$,	$a + b = b$	(4)
Substituting b for a,	$2b = b$	
Whence,	$2 = 1$	

Again, take the identical equation,

	$20 + 12 = 20 + 12$	(1)
Transposing,	$20 - 20 = 12 - 12$	(2)
Factoring,	$5(4-4) = 3(4-4)$	(3)
Whence,	$5 = 3$	(4)

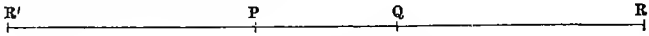
The members of equation (3) in each of the above examples are equal, because each member becomes 0, through the influence of a 0 factor; but it does not follow that the other factors are equal, as assumed in (4). In other words, the result of the division of each member of (3) is $\frac{0}{0}$, which may represent any finite quantity.

PROBLEM OF THE COURIERS.

185. The discussion of the following general problem, commonly known as that of Clairaut, will serve to further illustrate the significance of the anomalous forms considered in the last article.

PROBLEM.

Two couriers, A and B, are traveling along the same road, in the same direction, R′ R, at the rate of m and n miles respectively. If at any time, say 12 o'clock, A is at the point P, and B a miles from him at Q, *when* and *where* are they together?



Let $t =$ the required time in hours,

$x =$ the distance B travels in time t,

and $a + x =$ the distance A travels in time t.

But the distance each travels is equal to the rate of each per hour multiplied by the number denoting the time.

Therefore, $$mt = a + x \qquad (1)$$

Also, $$nt = x \qquad (2)$$

Subtracting (2) from (1), $$mt - nt = a \qquad (3)$$

Whence, $$t = \frac{a}{m-n} \qquad (4)$$

Substituting the value of t in (2), $$x = \frac{na}{m-n} \qquad (5)$$

It is proposed, now, to discuss these values on different suppositions; —

1. $m > n$.

This hypothesis makes the common denominator, $m-n$, in (4) and (5), positive; hence, the values of both t and x are positive. That is, the couriers are together *after* 12 o'clock, and at the *right* of P.

This interpretation corresponds with the conditions of the problem. For, A will eventually overtake B, and in advance of their position at 12 o'clock.

2. $m < n$.

This hypothesis makes the common denominator, $m-n$, negative; hence, both t and x are negative. Now, from what is known of negative results (Art. 181), these values

of t and x indicate that the couriers were together *before* 12 o'clock, and at the *left* of P.

This interpretation corresponds with the supposition made. For, A will never overtake B, but as they are traveling along the same road, they were together before they could have advanced as far as P.

3. $m = n$.

This hypothesis makes $m - n = 0$; then, we have

$$t = \frac{a}{0} = \infty, \text{ and } x = \frac{na}{0} = \infty.$$

Hence, the values of both t and x are infinity. That is, they will never be together.

This interpretation corresponds with the supposition made. For, if they travel at the same rate, and are a miles apart at 12 o'clock, they always *have preserved*, and always *will preserve*, the same distance.

4. $a = 0$, and $m > n$ or $m < n$.

Then, we have

$$t = \frac{0}{m-n} = 0, \text{ and } x = \frac{0}{m-n} = 0,$$

or, both the time and distance are nothing. That is, the couriers are together at 12 o'clock, at the point P, but at no other time or place.

This interpretation corresponds with the supposition made. For, since the distance between them at 12 o'clock is nothing, they are together then at P, but as their rates are unequal, they cannot be together after 12 o'clock, nor could they have been together before that time.

5. $a = 0$, and $m - n = 0$.

Then we have

$$t = \frac{0}{0}, \text{ and } x = \frac{0}{0},$$

or the values of t and x indeterminate. That is, the cou-

riers are together *at all times*, and *at any distance whatever* from P.

This interpretation corresponds with the supposition made. For, being together, and advancing at the same rate, they can never separate.

186. It may be remarked that the formula,

$$t = \frac{a}{m - n},$$

in the foregoing problem, is general, and may be applied to the solution of all similar

PROBLEMS.

1. A train of cars leaves Boston at 1 o'clock, P. M., at the rate of 24 miles an hour, and is followed by an express train at 3 o'clock, at the rate of 36 miles an hour. When will the latter overtake the former?

Here $a = 48$, $m = 36$, and $n = 24$;

whence, $t = \frac{48}{36 - 24} = 4$ hours;

and 3 o'clock + 4 hours = 7 o'clock, P. M., Ans.

2. At 12 o'clock both hands of a clock are together; when will they next be together?

Here $a = 12$, $m = 12$, and $n = 1$, hour spaces.

Whence, $t = \frac{12}{12 - 1} = 1$ h. $5\frac{5}{11}$ m., Ans.

3. At what time between 3 and 4 o'clock will the two hands of a clock be together? Ans. 3h. $16\frac{4}{11}$m.

4. Two men, A and B, set out from two places, distant from each other 144 miles, and travel in the same direction, along the same road. A goes 12 miles an hour, and B 4 miles an hour. In how many hours will they be together, and how far will each have traveled?

Ans. 18 hours; A 216 miles, and B 72 miles.

5. At what time between 12 and 1 o'clock do the hour and minute hands of a watch point in exactly opposite directions? Ans. $32\frac{8}{11}$ minutes after 12.

INEQUALITIES.

187. An INEQUALITY is an expression indicating that one of two quantities is greater or less than the other; as,

$$a > b, \text{ and } m < n.$$

The quantity on the left of the sign is called the *left member*, and that on the right, the *right member* of the inequality.

188. Two inequalities are said to subsist in *the same sense*, when the first member is the greater or less in both. Thus,

$$a > b \text{ and } c > d, \text{ or } 3 < 4 \text{ and } 2 < 3,$$

are inequalities which subsist in the same sense.

189. Two inequalities are said to subsist in *a contrary sense*, when the first member is the greater in the one, and the second in the other. Thus,

$$a > b \text{ and } c < d, \text{ or } x < y \text{ and } u > z,$$

are inequalities which subsist in a contrary sense.

190. In the discussion of inequalities, the terms *greater* and *less* must be taken as having an extended, or algebraic meaning. That is,

Of any two quantities, a *and* b, a *is the greater when* a − b *is positive, and* a *is the less when* a − b *is negative.*

Hence, a negative quantity may be considered as less than nothing, and, of two negative quantities, that is the greatest which has the least number of units. Thus,

$$0 > -2, \text{ and } -2 > -3.$$

191. *An inequality will continue in the same sense, after the same quantity has been added to, or subtracted from, each member.*

For, suppose $a > b$;

then, by Art. 190, $a - b$ is positive; consequently,

$$(a + c) - (b + c) \text{ and } (a - c) - (b - c)$$

are positive. Therefore,

$$a + c > b + c, \text{ and } a - c > b - c.$$

Hence, it follows that *a term may be transposed from one member of an inequality to the other, with its sign changed.*

192. *If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.*

For, to change all the signs is equivalent to transposing each term of the first member to the second, and each term of the second member to the first.

193. *If two or more inequalities subsisting in the same sense, be added, member to member, the resulting inequality will also subsist in the same sense.*

For, let

$$a > b,\ a' > b',\ a'' > b'', \ldots\ldots;$$

then, by Art. 190,

$$a - b,\ a' - b',\ a'' - b'', \ldots\ldots,$$

are all positive; consequently their sum,

$$a - b + a' - b' + a'' - b'', + \ldots\ldots,$$

or $(a + a' + a'' + \ldots\ldots) - (b + b' + b'' + \ldots\ldots)$,

is positive. Hence,

$$a + a' + a'' + \ldots\ldots > b + b' + b'' + \ldots\ldots$$

194. *If two inequalities subsisting in the same sense be subtracted, the one from the other, member from member, the resulting inequality will not always subsist in the same sense.*

For, let

$$a > b, \text{ and } a' > b';$$

then $a - b$ and $a' - b'$ are positive, but $a - b - (a' - b')$, or $a - a' - (b - b')$, may be either positive, negative, or 0; that is,

$$a - a' > b - b', \quad a - a' < b - b', \text{ or } a - a' = b - b'.$$

195. *An inequality will continue in the same sense, after each member has been multiplied or divided by the same positive quantity.*

For, suppose $a > b$;

then, since $a - b$ is positive, if m is positive,

$$m(a - b) \text{ and } \frac{1}{m}(a - b)$$

are positive. Hence,

$$ma > mb \text{ and } \frac{a}{m} > \frac{b}{m}.$$

196. *If each member of an inequality be multiplied or divided by the same negative quantity, the sign of inequality must be reversed.*

For, since multiplying or dividing by a negative quantity must change the signs of all the terms, the sign of inequality must be reversed (Art. 192).

197. The *solution* of an inequality consists in determining the limit to the value of its unknown quantity.

This may be done by the application of the preceding principles.

198. When, however, an inequality and an equation are given, containing two unknown quantities, the process of elimination will be required in the solution.

In verifying an inequality, if the symbols of the unknown quantities be taken equal to their respective limits, the inequality becomes an equation.

EXAMPLES.

1. Find the limit of x in the inequality,

$$7x - \frac{23}{3} > \frac{2x}{3} + 5.$$

Clearing of fractions, $21x - 23 > 2x + 15$

Transposing and uniting, $19x > 38$

Whence, $x > 2$

2. Find the limits of x in the inequalities,

$$ax + 5bx - 5ab > a^2,$$
$$bx - 7ax + 7ab < b^2.$$

Adding $5ab$ to both members of (1), and factoring,

$$(a + 5b)\,x > (a + 5b)\,a;$$

whence, $x > a.$

Subtracting $7ab$ from both members of (2), and factoring,

$$(b - 7a)\,x < (b - 7a)\,b;$$

whence, $x < b.$

3. Find the limits of x and y in the following inequality and equation,

$$\begin{cases} 4x + 6y > 52 & (1) \\ 4x + 2y = 32 & (2) \end{cases}$$

Subtracting (2) from (1), $4y > 20$

Whence, $y > 5$

Dividing (2) by 2, and transposing, $y = 16 - 2x$

Subst. value of y in (1), $4x + 96 - 12x > 52$

Or, $-8x > -44$

Whence, $x < 5\frac{1}{2}$

4. Given $5x - 6 > 19$, to find the limit of x.

Ans. $x > 5.$

5. Given $\left\{\begin{matrix} 2x - 5 > 25 \\ 3x - 7 < 2x + 13 \end{matrix}\right\}$, to find the limits of x.

Ans. $x > 15$ and $x < 20.$

6. Given $\left.\begin{matrix} 3x+1 > 13-x \\ 4x-7 < 2x+3 \end{matrix}\right\}$, to find an integral value of x.

Ans. $x = 4$.

7. Given $\left.\begin{matrix} 5x+3y > 46-y \\ 2y-2x = -8 \end{matrix}\right\}$, to find the limits of x and y.

Ans. $x > 6\frac{8}{9}$ and $y > 2\frac{8}{9}$.

8. Given $\left.\begin{matrix} \frac{cx}{3}+dx-cd > \frac{c^2}{3} \\ \frac{dx}{8}-cx+cd < \frac{d^2}{8} \end{matrix}\right\}$, to find the limits of x.

Ans. $x > c$ and $x < d$.

9. Given $\left.\begin{matrix} 2x+3y < 57 \\ 4x+2y = 64 \end{matrix}\right\}$, to find the limits of x and y.

Ans. $x > 9\frac{3}{4}$ and $y < 12\frac{1}{2}$.

10. A teacher being asked the number of his pupils, replied that twice their number diminished by 7 is greater than 29, and three times their number diminished by 5 is less than twice their number increased by 16. Required the number of his pupils. Ans. 19 or 20.

Here the inequalities give $x > 18$ and $x < 21$. That is, all numbers between 18 and 21 will satisfy the inequalities; but as, from the nature of the problem, the answer should be a whole number, the solutions are limited to $x = 19$, or $x = 20$.

11. Three times a certain number plus 16 is greater than twice that number plus 24, and two fifths of the number plus 5 is less than 11. Required the number.

Ans. $x > 8$ and $x < 15$.

That is, any number, whole or fractional, between 8 and 15.

12. A shepherd has a number of sheep such that three times the number increased by 2 exceeds twice the number increased by 61; and 5 times the number diminished by 70 is less than 4 times the number diminished by 9. How many sheep has he? Ans. 60.

INVOLUTION.

199. INVOLUTION is the process of raising a quantity to any required power.

This may be effected, as is evident from the definition of a power (Art. 19), by taking the given quantity as a factor as many times as there are units in the exponent of the required power. That is,

If the exponent is n, *the power is the product of* n *factors, when* n *is any entire quantity whatever.*

200. Any power of the product of two or more factors is equal to the product of the same power of each of the factors. Thus,

$$(a\,b)^n = a^n b^n.$$

For, $(a\,b)^n = a\,b \times a\,b \times a\,b \ldots\ldots$ to n factors, is the same as $a \times a \times a \ldots\ldots$ to n factors $\times\ b \times b \times b \ldots\ldots$ to n factors, or $a^n \times b^n$, and the same reasoning may be employed when there are more than two factors. That is,

The n*th power of the product of two or more factors is equal to the product of the* n*th power of each of the factors.*

201. The power of any power of a quantity is equal to that expressed by the product of the exponents of the two powers. Thus,

$$(a^n)^m = a^{mn}.$$

For, if we form the product of m factors of which each is a^n, the result, by the rule of multiplication (Art. 67), is a^{mn}. That is,

The m*th power of the* n*th power of any quantity is equal to the* mn*th power of that quantity.*

202. If the quantity to be involved is positive, the sign of all its powers will be positive; but if the quantity is negative, all its even powers will be positive, and all its odd powers negative. Thus,

$$a \times a = a^2$$
$$a \times a \times a = a^2 \times a = a^3, \text{ and so on.}$$

For, any positive factor taken any number of times must give a positive result (Art. 66).

Again,

$$(-a) \times (-a) = +a^2$$
$$(-a) \times (-a) \times (-a) = +a^2 \times (-a) = -a^3, \text{ and so on.}$$

For, a negative multiplier causes the sign of the product to be the opposite of that of the multiplicand (Art. 66), and therefore each additional negative factor changes the sign of the result. So that, since n may be any quantity whatever, in the expression,

$$(-a)^n = \pm a^n,$$

by the double sign $\pm$, is to be understood, if the nth power of $-a$ is even, it has the positive sign, but if odd, the negative sign. Hence,

Every EVEN *power is positive, and every* ODD *power has the same sign as its root.*

POWERS OF MONOMIALS.

203. From the preceding principles and the rule for multiplication, we have, for raising a monomial to any power, the following

RULE.

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power, making the sign of every even power positive, and the sign of every odd power the same as that of its root.

EXAMPLES.

1. Required the third power of $-3a^2b$.

$$(-3a^2b)^3 = -3^3a^6b^3 = -27a^6b^3.$$

2. Raise a^2x to the second power. Ans. a^4x^2.
3. Raise $-ab^2c^3$ to the fourth power. Ans. $a^4b^8c^{12}$.
4. Required the mth power of a^nb. Ans. $a^{mn}b^m$.
5. Required the fourth power of $2x^m$. Ans. $16x^{4m}$.
6. Find the fifth power of $2ab^2x^3$. Ans. $32a^5b^{10}x^{15}$.
7. Find the nth power of a^2b^2. Ans. $a^{2n}b^{2n}$.
8. Find the value of $(-a^2c^3)^3$. Ans. $-a^6c^9$.
9. Find the value of $(3a^nb^mc^2)^0$. Ans. 1.
10. Find the value of $(-2ab^nx)^5$. Ans. $-32a^5b^{5n}x^5$.
11. Find the value of $(-11bxy^2)^m$. Ans. $\pm 11^m b^m x^m y^{2m}$.

204. From the definition of involution (Art. 199), and the rule for multiplication of fractions (Art. 137), it follows that a fraction is involved *by raising both the numerator and the denominator to the required power.*

12. Required the second power of $\frac{ac}{b}$. Ans. $\frac{a^2c^2}{b^2}$.
13. Required the third power of $\frac{3a^2b^3}{4xy^4}$. Ans. $\frac{27a^6b^9}{64x^3y^{12}}$.

14. Find the square of $-\frac{2ax^2}{3b}$. Ans. $\frac{4a^2x^4}{9b^2}$.

15. Find the sixth power of $\frac{2}{3}a^3x^2$. Ans. $\frac{64}{729}a^{18}x^{12}$.

16. Find the third power of $-\frac{2xy^2}{3b}$. Ans. $-\frac{8x^3y^6}{27b^3}$.

17. Find the value of $\left(\frac{-bcx^n}{4a^2}\right)^5$. Ans. $\frac{-b^5c^5x^{5n}}{1024a^{10}}$.

205. The rule already given (Art. 203) holds when the quantity to be involved has negative exponents, and when the exponent of the required power is negative.

For, $(a^{-n})^m = \left(\frac{1}{a^n}\right)^m = \frac{1}{a^{mn}} = a^{-mn}$,

and $(a^n)^{-m} = \frac{1}{(a^n)^m} = \frac{1}{a^{mn}} = a^{-mn}$ (Art. 77).

18. Required the third power of $2a^{-2}$. Ans. $8a^{-6}$.

19. Required the fourth power of $-3m^{-3}$.
Ans. $81m^{-12}$.

20. Find the cube of $-2x^ny^{-m}$. Ans. $-8x^{3n}y^{-3m}$.

21. Find the value of $(-3a^2b^3c^4d^{-1})^6$.
Ans. $729a^{12}b^{18}c^{24}d^{-6}$.

22. Find the value of $(-3x^2y^{-2}z^{-1})^{-3}$.
Ans. $-\frac{1}{27}x^{-6}y^6z^3$.

23. Raise $-a^{-n}b^mc^{-2}$ to the mth power.
Ans. $\pm a^{-mn}b^{m^2}c^{-2m}$.

24. Raise $\frac{-2a^{-m}x^2}{bc^{-n}y^2}$ to the fourth power.
Ans. $16a^{-4m}b^{-4}c^{4n}x^8y^{-8}$.

25. Raise $-8a^2bx^{-1}y^{-3}$ to the -2d power.
Ans. $\frac{1}{64}a^{-4}b^{-2}x^2y^6$.

POWERS OF POLYNOMIALS.

206. Polynomials may be raised to any power, it is obvious from Art. 199, by the process of successive multiplications. Thus,

$$(a+b)^2 = (a+b)\,(a+b) = a^2+2\,a\,b+b^2,$$
$$(a+b)^3 = (a+b)\,(a+b)\,(a+b) = a^3+3\,a^2\,b+3\,a\,b^2+b^3,$$

and so on. Hence the following

General Rule.

Multiply the given quantity by itself, until it has been taken as a factor as many times as there are units in the exponent of the power.

Examples.

1. Find the third power of $a - b$.

   Ans. $a^3 - 3\,a^2\,b + 3\,a\,b^2 - b^3$.

2. Find the second power of $\frac{a}{b} - \frac{b}{a}$.

   Ans. $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$.

3. Cube $1 + a^2 + b^{-2}$.

   Ans. $1 + 3\,a^2 + 3\,b^{-2} + 3\,a^4 + 6\,a^2\,b^{-2} + 3\,b^{-4} + a^6 + 3\,a^4\,b^{-2} + 3\,a^2\,b^{-4} + b^{-6}$.

4. Raise $a + m - n$ to the second power.

   Ans. $a^2 + 2\,a\,m - 2\,a\,n + m^2 - 2\,m\,n + n^2$.

5. Expand $(a^m - a^n)^4$.

   Ans. $a^{4m} - 4\,a^{3m+n} + 6\,a^{2m+2n} - 4\,a^{m+3n} + a^{4n}$.

6. Expand $(a + b)^5$.

   Ans. $a^5 + 5\,a^4\,b + 10\,a^3\,b^2 + 10\,a^2\,b^3 + 5\,a\,b^4 + b^5$.

POLYNOMIAL SQUARES.

207. It has been shown (Art. 90) that the *square* of any binomial expression can be written down, without recourse to formal multiplication, by application of the formulas for the square of $a + b$ and $a - b$, or

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

It may likewise be shown by actual multiplication, that

$$(a + b + c)^2 = a^2 + 2a(b + c) + b^2 + 2bc + c^2$$
$$(a + b + c + d)^2 = a^2 + 2a(b + c + d) + b^2 + 2b(c + d) + c^2 + 2cd + d^2,$$

and so on.

These results, for convenience of enunciation, may take another form,

$$(a + b)^2 = a^2 + b^2 + 2ab$$
$$(a - b)^2 = a^2 + b^2 - 2ab$$
$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$
$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd,$$

and so on. Hence the following

RULE.

Write the square of each term, together with twice its product by each of the terms following it.

EXAMPLES.

1. Square $a - b + c$.

   Ans. $a^2 - 2ab + 2ac + b^2 - 2bc + c^2$.

2. Square $2x^2 + 3x + 4$.

   Ans. $4x^4 + 12x^3 + 25x^2 + 24x + 16$.

3. Square $2x^2 - 3x + \frac{1}{2}$.

Ans. $4x^4 - 12x^3 + 11x^2 - 3x + \frac{1}{4}$.

4. Square $a - b - c + d$.

Ans. $a^2 - 2ab - 2ac + 2ad + b^2 + 2bc - 2bd + c^2 - 2cd + d^2$.

5. Square $x^3 + 2x^2 + x + 2$.

Ans. $x^6 + 4x^5 + 6x^4 + 8x^3 + 9x^2 + 4x + 4$.

6. Expand $(1 - 2x + 3x^2)^2$.

Ans. $1 - 4x + 10x^2 - 12x^3 + 9x^4$.

7. Expand $(1 + x + x^2 + x^3)^2$.

Ans. $1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6$.

8. Required the square of $a + b + c + d + e$.

Ans. $a^2 + 2ab + 2ac + 2ad + 2ae + b^2 + 2bc + 2bd + 2be + c^2 + 2cd + 2ce + d^2 + 2de + e^2$.

POLYNOMIAL CUBES.

208. The *cube* of $a + b$, and of $a - b$, readily obtained by multiplication, give the formulas,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$

which, for convenience of enunciation, may be put under the form,

$$(a + b)^3 = a^3 + b^3 + 3ab\,(a + b),$$
$$(a - b)^3 = a^3 - b^3 - 3ab\,(a - b).$$

That is,

The CUBE *of a* BINOMIAL *is equal to the cube of the two terms, together with three times the product of the two terms multiplied by each of the terms.*

But it may be also shown by actual multiplication, that

$$(a+b+c)^3 = a^3 + 3a^2(b+c) + b^3 + 3b^2(a+c) + c^3 + 3c^2(a+b) + 6abc,$$

$$(a+b+c+d)^3 = a^3 + 3a^2(b+c+d) + b^3 + 3b^2(a+c+d) + c^3 + 3c^2(a+b+d) + d^3 + 3d^2(a+b+c) + 6abc + 6abd + 6acd + 6bcd,$$

and so on. Hence, for finding the cube of expressions containing three or more terms, the following

RULE.

Write the cube of each term, together with three times the product of its square by each of the other terms, and also six times the product of every three different terms.

EXAMPLES.

1. Find the third power of $a + b - c$.

   Ans. $a^3 + b^3 - c^3 + 3a^2b - 3a^2c + 3ab^2 - 3cb^2 + 3ac^2 + 3bc^2 - 6abc.$

2. Find the third power of $x^2 - x - 1$.

   Ans. $x^6 - 3x^5 + 5x^3 - 3x - 1.$

3. Required the third power of $a - b + 1$.

   Ans. $a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 3a - b^3 + 3b^2 - 3b + 1.$

4. Required the third power of $1 + x + x^2 + x^3$.

   Ans. $1 + 3x + 6x^2 + 10x^3 + 12x^4 + 12x^5 + 10x^6 + 6x^7 + 3x^8 + x^9.$

5. What is the third power of $a - b - 2c^2 - d^3$?

   Ans. $a^3 - b^3 - 8c^6 - d^9 - 3a^2b - 6a^2c^2 - 3a^2d^3 + 3ab^2 - 6b^2c^2 - 3b^2d^3 + 12ac^4 - 12bc^4 - 12c^4d^3 + 3ad^6 - 3bd^6 - 6c^2d^6 + 12abc^2 + 6abd^3 + 12ac^2d^3 - 12bc^2d^3.$

EVOLUTION.

209. Evolution is the process of extracting any required root of a quantity.

This may be effected, as is evident from the definition of a root (Art. 19), by determining a quantity which, when raised to the proposed power, will produce the given quantity. It is, therefore, the reverse of involution.

210. Any quantity whose root can be extracted is called a *perfect power*, and any quantity whose root can not be extracted is called an *imperfect power*.

A quantity, however, may be a perfect power of one degree, and not of another.

Thus, 8 is a perfect cube, but not a perfect square.

211. To extract any root of a simple quantity, the exponent of that quantity must be divided by the index of the root.

For, since the nth power of a^m is a^{mn} (Art. 201),

$$\sqrt[n]{a^{mn}} = a^m.$$

212. Any root of the product of two or more factors is equal to the product of the same root of each of the factors.

For, it has been shown, in raising a quantity composed of factors to any required power, that each of the factors is raised to the same power (Art. 200). Thus, if n be equal to any entire number whatever,

$$(a\,b\,.\,.\,.\,.)^n = a^n\,b^n\,.\,.\,.\,.;$$

and, conversely (Art. 209),

$$\sqrt[n]{a\,b\,.\,.\,.\,.} = \sqrt[n]{a} \times \sqrt[n]{b}\,.\,.\,.\,.,$$

or, $$(a\,b\,.\,.\,.\,.)^{\frac{1}{n}} = a^{\frac{1}{n}}\,b^{\frac{1}{n}}\,.\,.\,.\,.;$$ that is,

The n*th root of the product of two or more factors is equal to the product of the* n*th root of each of the factors.*

It likewise follows that

The product of the n*th root of two or more factors is equal to the* n*th root of their product.*

213. From the relation of a root to its corresponding power, it follows, from Art. 202, that

1. *The odd roots of any quantity have the same sign as the power.*

For, a positive quantity raised to any power is positive; but a negative quantity raised to any odd power is negative. Thus,

$$\sqrt[3]{a^3} = a, \text{ or } \sqrt[5]{-a^5} = -a.$$

2. *The even roots of a positive quantity are either positive or negative.*

For, either a positive or a negative quantity raised to an even power is positive. Thus,

$$\sqrt{a^2} = \pm a, \text{ or } \sqrt[4]{a^4} = \pm a.$$

3. *Even roots of a negative quantity are not possible.*

For, no quantity raised to an even power can produce a negative result. Such roots are called *impossible*, or *imaginary* quantities.

SQUARE ROOT OF NUMBERS.

214. A Perfect Square is any number or quantity that can be resolved into two equal factors (Art. 199); and, consequently, has an exact square root (Art. 210).

215. If a number be separated into as many groups, or periods, of two figures each, as is possible, from right to left, these groups will correspond, respectively, to the *units*, *tens*, *hundreds*, etc., in its square root.

For, the first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10;

and their squares are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Also, the square of 99 is 9801, of 100 is 10000, of 999 is 998001, of 1000 is 1000000, and so on. Hence,

If a point be placed over every second figure in any integral number, beginning with the units' place, the points will show the number and order of the figures in its square root.

216. Every number, expressed by more than one place of figures, can be separated into two parts, tens and units.

Therefore, if we denote the tens of a number by a, and the units by b, the number will be denoted by $a + b$, and its square by

$$(a + b)^2 = a^2 + 2ab + b^2 = a^2 + (2a + b)\, b.$$

Then, by this formula, if $a = 3$ tens, or 30, and $b = 6$, we have

$$3 \text{ tens} + 6 \text{ units} = 30 + 6 = 36;$$

and $$36^2 = (30 + 6)^2 = 30^2 + (60 + 6) \times 6 = 1296.$$

Hence,

Every number expressed by more than two places of figures is equal to the square of the tens in its ROOT, *plus the product of twice the tens plus the units, multiplied by the units.*

1. Let it be required to find the square root of 4356.

OPERATION.

$$\left.\begin{array}{r|l} 4\,\dot{3}\,5\,\dot{6} & 60 + 6 \\ 3\,6\,0\,0 & \\ \hline 756 & \\ 756 & \\ \hline \end{array}\right\} \quad 120 + 6$$

or

$$\left\{\begin{array}{r|l} 4\,\dot{3}\,5\,\dot{6} & 66 \\ 36 & \\ \hline 756 & \\ 756 & \\ \hline \end{array}\right. \quad 126$$

The root of the given number, according to Art. 215, consists of two places of figures.

Let, now, $a + b$ denote the root, where a is the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of ten whose square is less than 4300; this we find to be 60. Subtracting a^2, that is, the square of 60, from the given number, we have the remainder 756. Dividing this remainder by $2a$, that is, by 120, gives 6, which is the value of b. Then $(2a + b)\, b$, that is, 126×6, or 756, is the quantity to be subtracted; and as there is now no remainder, we conclude that $60 + 6$, or 66, is the required square root.

Had the root consisted of three places of figures, we could have let a represent the hundreds, and b the tens; then, having obtained a and b as before, we might let the hundreds and tens together be considered as a new value of a, and find a new value of b for the units. Hence the following

RULE.

Separate the given number into periods, by pointing every second figure, beginning with the units' place.

Find the greatest square in the left-hand period, and place its root on the right; subtract the square of this root from the first period, and to the remainder bring down the next period for a dividend.

Divide this quantity, omitting the last figure, by double the part of the root already found, and annex the result to the root, and also to the divisor.

Multiply the divisor as it now stands by the part of the root last obtained, and subtract the product from the dividend.

If there are more periods to be brought down, continue the operation in the same manner as before.

If there be a final remainder, the given number has not an exact root; and, since the rule applies equally to

decimals, we may continue the operation, by annexing periods of decimal ciphers, and thus obtain a decimal part, to be added to the integral part of the root already found.

It will be observed that decimals require to be pointed to the right; and if they have not an exact root, we may continue to form decimal periods, and obtain decimal figures in the root to any desirable extent.

As the trial divisor is necessarily an *incomplete* divisor, it is sometimes found that after completion it gives a product larger than the dividend. In such a case, the figure found for the root is too large, and one less must be substituted for it.

The root of a common fraction may be found, it is evident from Art. 204, by taking the root of both the numerator and denominator.

EXAMPLES.

2. Extract the square root of 273529. Ans. 523.
3. Extract the square root of 45796. Ans. 214.
4. Extract the square root of 106929. Ans. 327.
5. Extract the square root of 33.1776. Ans. 5.76.
6. Extract the square root of .9409. Ans. .97.
7. Extract the square root of $\frac{6}{9}\frac{5}{0}\frac{6}{2}\frac{1}{5}$. Ans. $\frac{8}{9}\frac{1}{5}$.
8. Extract the square root of $5\frac{11}{49}$. Ans. $2\frac{3}{7}$.
9. Extract the square root of 2 to three decimal places. Ans. 1.414 +.

Here, the sign + is used to denote that the approximate root is less than the true root.

217. To find the approximate root of an imperfect square, within a given fractional unit, we may

Multiply the given number by the square of the denominator of the fraction determining the degree of exactness; then extract

the square root of the product to the nearest unit, and divide the result by the denominator of the fraction.

For, let a be any imperfect square, of which it is required to find the square root to within a fraction $\frac{1}{n}$; then, since a may be put under the form of $\frac{a}{1}$, by multiplying both terms of the expression by n^2, we have

$$a = \frac{a n^2}{n^2}.$$

Denote the entire part of the square root of $a n^2$ by r, and $a n^2$ will be comprised between

$$r^2 \text{ and } (r+1)^2;$$

consequently $\frac{a n^2}{n^2}$ will be comprised between

$$\frac{r^2}{n^2} \text{ and } \frac{(r+1)^2}{n^2};$$

hence the square root of $\frac{a n^2}{n^2}$, that is of a, will be comprised between

$$\frac{r}{n} \text{ and } \frac{r+1}{n}.$$

But the difference between $\frac{r}{n}$ and $\frac{r+1}{n}$ is $\frac{1}{n}$, whence $\frac{r}{n}$ will be the root of a to within $\frac{1}{n}$.

10. Find the square root of 59 to within $\frac{1}{12}$.

$59 \times (12)^2 = 8496$; $\sqrt{8496} = 92+$; $92 \div 12 = 7\frac{8}{12}$, Ans.

11. Find the square root of 7 to within $\frac{1}{10}$.

Ans. $2\frac{6}{10}$.

12. Find the square root of 11 to within $\frac{1}{15}$.

Ans. $3\frac{5}{15}$.

13. Find the square root of 12 to within .01.

Ans. 3.46.

14. Find the square root of 5 to within .00001.

Ans. 2.23607.

218. To find the approximate square root of a common fraction, whose denominator, only, is a perfect square, we may, obviously,

Take the approximate square root of the numerator, and divide it by the square root of the denominator.

15. Find the square root of $\frac{15}{25}$ to within $\frac{1}{5}$.

$$\frac{\sqrt{15}}{\sqrt{25}} = \frac{4}{5}, \quad \text{Ans.}$$

16. Find the square root of $\frac{5}{9}$ to three decimal places.

$$\frac{\sqrt{5}}{\sqrt{9}} = \frac{2.236+}{3} = .745, \quad \text{Ans.}$$

17. Find the square root of $\frac{2}{49}$ to five places of decimals. Ans. .20203.

219. The fraction may be changed to a decimal when its denominator is not a perfect square; or we may multiply both terms of the fraction by any number which will make its denominator a perfect square, and proceed as in the last Article.

18. Find the square root of $\frac{9}{15}$ to within $\frac{1}{15}$.

$$\frac{9}{15} \times \frac{15}{15} = \frac{135}{225}; \quad \frac{\sqrt{135}}{\sqrt{225}} = \frac{12}{15}.$$

19. Find the square root of $\frac{3}{5}$ to within $\frac{1}{5}$, and to within $\frac{1}{60}$.

$$\left.\begin{aligned} &\frac{3}{5} \times \frac{5}{5} = \frac{15}{25}; \quad \frac{\sqrt{15}}{\sqrt{25}} = \frac{4}{5}; \\ &\frac{15}{25} \times \left(\frac{12}{12}\right)^2 = \frac{2160}{3600}; \quad \frac{\sqrt{2160}}{\sqrt{3600}} = \frac{46}{60}. \end{aligned}\right\} \text{Ans.}$$

20. Find the square root of $\frac{5}{6}$ to within $\frac{1}{48}$.

Ans. $\frac{44}{48}$.

21. Find the square root of $\frac{11}{17}$ to four decimal places.

Ans. .8044.

CUBE ROOT OF NUMBERS.

220. A PERFECT CUBE is any number or quantity that can be resolved into three equal factors (Art. 199); and, consequently, has an exact cube root (Art. 210).

221. If a number be separated into as many groups, or periods, of three figures each, as is possible, from right to left, these groups will correspond, respectively, to the *units*, *tens*, *hundreds*, etc., in its cube root.

For, the first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10;

and their cubes are

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

Also, the cube of 99 is 970299, of 100 is 1000000, of 999 is 997002999, of 1000 is 1000000000, and so on. Hence,

If a point be placed over every third figure in any integral number, beginning with the units' place, the points will show the number and order of the figures in its cube root.

222. Since every number expressed by more than one place of figures, can be separated into two parts, tens and units,

If we denote the tens of a number by a, and the units by b, the number will be denoted by $a + b$, and its cube by

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b.$$

Then, by this formula, if $a = 7$ tens, or 70, and $b = 4$, we have

$$7 \text{ tens} + 4 \text{ units} = 70 + 4 = 74,$$

and

$$74^3 = (70+4)^3 = 70^3 + (14700 + 840 + 16) \times 4 = 405224.$$

Hence,

Every number expressed by more than three places of figures is equal to the cube of the tens in its root, plus the sum of three times the square of the tens, three times the product of the tens and units, and the square of the units, multiplied by the units.

1. Let it be required to find the cube root of 405224.

OPERATION.

$$
\begin{array}{rl}
 & 40\dot{5}2\dot{2}\dot{4} \mid 74 \\
 & 343 \\
\hline
14700 & 62224 \\
840 & \\
16 & \\
\hline
15556 \times 4 = & 62224
\end{array}
$$

The root of the given number, according to Article 221, consists of two places of figures.

Let, now, $a + b$ denote the root, where a is the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of ten whose cube is less than 405000; this we find to be 70, or 7 tens. Subtracting a^3, that is the cube of 70, or 343000, from the given number, we have the remainder 62224. Dividing this remainder by $3a^2$, that is, by three times the square of 70, or 14700, we obtain the value of b, or 4. Then $(3a^2 + 3ab + b^2)\, b$, that is, $(14700 + 840 + 16)\, 4$, or $15556 \times 4 = 62224$, is the quantity to be subtracted; and as there is now no remainder, we conclude that $70 + 4$, or 74, is the required root.

For brevity, instead of writing 70 in the root, we may simply write 7 in the tens' place; and for the cube of 70 write the cube of 7, or 343, without the ciphers, under the proper period.

Had the root consisted of three figures, we could have let a represent the hundreds, and b the tens; then, having obtained a and b as before, we might let the hundreds and tens together be considered as a new value of a, and find a new value of b for the units. Hence the following

RULE.

Separate the given number into periods, by pointing every third figure, beginning with the units' place.

Find the greatest cube in the left-hand period, and place its root on the right; subtract the cube of this root from the first period, and to the remainder bring down the next period for a dividend.

At the left of the dividend write three times the square of the root already found, for a trial divisor; divide the dividend, omitting the last two figures, by it, and write the quotient for the next figure of the root.

Add together the trial divisor, with two ciphers annexed; three times the product of the last root figure by the rest of the root, with one cipher annexed; and the square of the last root figure; and the sum will be the complete divisor.

Multiply the complete divisor by the last root figure, and subtract the product from the dividend.

If there are more periods to bring down, continue the operation in the same manner as before.

The observations made under the rule for the extraction of the square root (Art. 216) are equally applicable to the extraction of the cube root.

EXAMPLES.

2. Extract the cube root of 8.144865728.

OPERATION.

$$
\begin{array}{rl|l}
 & \dot{8}.1\,4\,4\,\dot{8}\,6\,\dot{5}\,7\,2\,\dot{8} & 2.0\,1\,2 \\
 & 8 & \\
\hline
1\,2\,0\,0\,0\,0 & 1\,4\,4\,8\,6\,5 & \\
6\,0\,0 & & \\
\quad 1 & & \\
\hline
1\,2\,0\,6\,0\,1 \times 1 = & 1\,2\,0\,6\,0\,1 & \\
\hline
1\,2\,1\,2\,0\,3\,0\,0 & 2\,4\,2\,6\,4\,7\,2\,8 & \\
1\,2\,0\,6\,0 & & \\
\quad 4 & & \\
\hline
1\,2\,1\,3\,2\,3\,6\,4 \times 2 = & 2\,4\,2\,6\,4\,7\,2\,8 &
\end{array}
$$

Here, it will be observed that, in consequence of the 0 in the root, we annex two additional ciphers to the trial divisor, 1200, and bring down to the corresponding dividend another period.

3. Extract the cube root of 1860867. Ans. 123.
4. Extract the cube root of .724150792. Ans. .898.
5. Extract the cube root of 1481.544. Ans. 11.4.
6. Extract the cube root of $\frac{2\,9\,7\,9\,1}{6\,8\,1\,4\,7\,2}$. Ans. $\frac{3\,1}{6\,8}$.
7. Extract the cube root of $\frac{3}{17}$ to four decimal places. Ans. .5609+.

Here, it is convenient to change the given fraction to a decimal, and then take the required root.

ROOTS OF MONOMIALS.

223. From the principles contained in Articles 209–213, we obtain the following

RULE.

Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root, making

the sign of every even root of a positive quantity $\pm$, *and the sign of every odd root of any quantity the same as that of the quantity.*

If the given quantity is a fraction, it follows, from Art. 204, that we may take the square root of both its terms for the required root.

EXAMPLES.

1. Find the square root of $9\,a^4\,b^2\,c^6$.

OPERATION.

$$\sqrt{9\,a^4\,b^2\,c^6} = 9^{\frac{1}{2}} \times a^{\frac{4}{2}} \times b^{\frac{2}{2}} \times c^{\frac{6}{2}} = \pm\, 3\,a^2\,b\,c^3, \text{ Ans.}$$

2. Find the cube root of $-125\,x^3\,y^6$. Ans. $-5\,x\,y^2$.

3. Find the fourth root of $81\,a^4\,b^8$. Ans. $\pm\, 3\,a\,b^2$.

4. Find the fifth root of $\frac{32\,m^5\,n^{10}}{243}$. Ans. $\frac{2\,m\,n^2}{3}$.

5. Find the square root of $9\,a^4\,b^2\,c^{-12}$.

Ans. $\pm\, 3\,a^2\,b\,c^{-6}$.

6. Find the square root of $625\,a^{12}\,c$.

Ans. $\pm\, 25\,a^6\,c^{\frac{1}{2}}$.

Here, as we can not extract the root of c, its root is indicated in the result by the fractional exponent.

7. Find the mth root of $a\,b^{-mn}$. Ans. $a^{\frac{1}{m}}\,b^{-n}$.

8. Find the cube root of $-8\,a^{-3}\,b^6\,x^{-2}$.

Ans. $-2\,a^{-1}\,b^2\,x^{-\frac{2}{3}}$.

9. Find the value of $(16\,x^{2m+2}\,a^{-2n})^{\frac{1}{2}}$.

Ans. $\pm\, 4\,x^{m+1}\,a^{-n}$.

10. Required the value of $(3\,a^m\,b^{2n}\,d^2)^{\frac{1}{n}}$.

Ans. $3^{\frac{1}{n}}\,a^{\frac{m}{n}}\,b^2\,d^{\frac{2}{n}}$.

11. Required the mth root of $\frac{1}{a^n}(x-y)^{\frac{m}{n}}$.

Ans. $a^{-\frac{n}{m}}(x-y)^{\frac{1}{n}}$.

12. Required the square root of $(a+x)^2\, b^2\, x\, y^4$.

Ans. $\pm\,(a+x)\, b\, x^{\frac{1}{2}}\, y^2$.

SQUARE ROOT OF POLYNOMIALS.

224. The manner of forming the square of a polynomial must, by reversing the process, lead to the discovery of its root. If we take any binomial, as $a+b$, we have

$$(a+b)^2 = a^2 + 2ab + b^2;$$

and the last two terms of this expression factored give

$$(2a+b)\, b.$$

1. Let us now reverse the involution, and discover how the root $a+b$ may be derived from the square $a^2+2ab+b^2$.

OPERATION.

$a^2 + 2ab + b^2$	$a+b$
a^2	
$2a+b$ | $2ab+b^2$	
$2ab+b^2$	

The square root of the first term, a^2, is a, which is the first term of the required root. Subtracting the square of a from the given polynomial, we have $2ab+b^2$, or $(2a+b)\, b$, for a remainder or dividend. Dividing the first term of the dividend, $2ab$, by $2a$, which is double the first term of the root, we obtain b, the other term of the root, which, connected to $2a$, completes the divisor, $2a+b$. Multiplying this divisor by the last term of the root, b, and subtracting the product, $2ab+b^2$, from the dividend, we have no remainder.

By a like process, a root consisting of more than two

terms may be found from its square, since all such roots can be expressed in a binomial form. Thus,

$$a + b + c = (a + b) + c,$$

and its square,

$$a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = (a+b)^2 + 2(a+b)c + c^2,$$

which, factored, gives

$$(a + b + c)^2 = a^2 + (2a + b)b + (2a + 2b + c)c.$$

Hence the following

RULE.

Arrange the terms according to the powers of some letter.

Find the square root of the first term, write it as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by double the part of the root already found, and annex the result to the root, and also to the divisor.

Multiply the divisor as it now stands by the term of the root last obtained, and subtract the product from the dividend.

If there are other terms remaining, continue the operation in the same manner as before.

Since all even roots have the double sign $\pm$ (Art. 223), the square root obtained by the rule will remain a root when all the signs are changed. In fact, in the example above, the first term in the root might have been found by taking $-a$ as the root of a^2, and continuing the operation in the same manner, we should have $-a-b$ for the root.

EXAMPLES.

2. Find the square root of

$$9x^4 - 12x^3 + 16x^2 - 8x + 4.$$

OPERATION.

$$
\begin{array}{r|l}
9x^4 - 12x^3 + 16x^2 - 8x + 4 & 3x^2 - 2x + 2 \\
9x^4 & \\
\hline
6x^2 - 2x \;\Big|\; -12x^3 + 16x^2 & \\
\quad\;\; -12x^3 + 4x^2 & \\
\hline
6x^2 - 4x + 2 \;\Big|\; 12x^2 - 8x + 4 & \\
\quad\;\; 12x^2 - 8x + 4 & \\
\hline
\end{array}
$$

Here, it will be observed that the successive subtrahends in the operation, $9x^4$, $(6x^2 - 2x)(-2x)$, and $(6x^2 - 4x + 2)\,2$, in the aggregate equal $(3x^2 - 2x + 2)^2$, in accordance with the formula upon which the rule is based.

3. Find the square root of $4x^4 - 4x^3 - 3x^2 + 2x + 1$.

Ans. $2x^2 - x - 1$.

4. Find the square root of $4a^4 - 16a^3 + 24a^2 - 16a + 4$.

Ans. $2a^2 - 4a + 2$.

5. Find the square root of $m^2 + 2m - 1 - \frac{2}{m} + \frac{1}{m^2}$.

Ans. $m + 1 - \frac{1}{m}$.

6. Extract the square root of $(x + x^{-1})^2 - 4(x - x^{-1})$.

Ans. $x - x^{-1} - 2$.

225. No *rational binomial* is an exact square; but, by the rule, the *approximate root* may be found.

7. Find the approximate square root of $1 + x$ to three terms.

Ans. $1 + \frac{x}{2} - \frac{x^2}{8} + \text{etc.}$

8. Find the approximate square root of $a + b$ to three terms.

Ans. $a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} - \frac{b^2}{8a^{\frac{3}{2}}} + \text{etc.}$

9. Find the approximate square root of $1 - x$ to four terms.

Ans. $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} -$ etc.

10. Find the approximate square root of $a^2 + x^2$ to five terms.

Ans. $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} +$ etc.

226. When a trinomial is a complete square, two of the terms are squares, and the other term is twice the product of their square roots (Art. 90); hence, if such a trinomial be arranged according to the powers of either of its letters,

The square roots of its extreme terms, united by the sign of its middle term, will be the square root of the trinomial.

11. Find the square root of $9y^4 - 6x^3y^2 + x^6$.

Ans. $3y^2 - x^3$.

12. Find the square root of $a^2 - ax + \frac{x^2}{4}$.

Ans. $a - \frac{x}{2}$.

13. Find the square root of $a^{2m} - 4a^{m+n} + 4a^{2n}$.

Ans. $a^m - 2a^n$.

14. Find the square root of $\frac{a^2}{b^2} - \frac{4ab}{3bc} + \frac{4b^2}{9c^2}$.

Ans. $\frac{a}{b} - \frac{2b}{3c}$.

CUBE ROOT OF POLYNOMIALS.

227. An investigation of the formation of a polynomial cube, by reversing the process, must lead to the discovery of its root. If we take any binomial, as $a + b$, we have

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

and the last three terms of this expression, factored, give

$$(3a^2 + 3ab + b^2)\, b.$$

1. Let us now, by reversing the involution, discover how the root $a + b$ may be derived from the cube $a^3 + 3a^2b + 3ab^2 + b^3$.

OPERATION.

$$\begin{array}{r|l} a^3 + 3a^2b + 3ab^2 + b^3 & a + b \\ a^3 & \\ \hline 3a^2 + 3ab + b^2 \;\Big|\; 3a^2b + 3ab^2 + b^3 & \\ \qquad\qquad\quad\; 3a^2b + 3ab^2 + b^3 & \\ \hline \end{array}$$

The cube root of the first term, a^3, is a, which is the first term of the required root. Subtracting the cube of a from the given polynomial, we have $3a^2b + 3ab^2 + b^3$, or $(3a^2 + 3ab + b^2)\,b$, for a remainder or dividend. Dividing the first term of the dividend by the trial divisor, $3a^2$, which is three times the square of the first term of the root, we obtain b, the other term of the root. Adding, now, to the trial divisor, $3a^2$, three times the product of the first term of the root by the last, and the square of the last term of the root, we have for the complete divisor, $3a^2 + 3ab + b^2$. Multiplying this by b, the last term of the root, and subtracting the product from the dividend, the whole cube of $a + b$ has been subtracted.

By a like process, a root of more than two terms may be found from its cube, since all such roots can be expressed in a binomial form. Thus,

$$a + b + c = (a + b) + c,$$

and its cube,

$$\begin{aligned} a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 \\ = (a + b)^3 + 3(a + b)^2 c + 3(a + b)c^2 + c^3, \end{aligned}$$

which, factored, gives

$$\begin{aligned} (a + b + c)^3 = a^3 + (3a^2 + 3ab + b^2)\,b + (3a^2 + 6ab \\ + 3b^2 + 3ac + 3bc + c^2)\,c. \end{aligned}$$

Hence the following

RULE.

Arrange the terms according to the powers of some letter.

Find the cube root of the first term, write it as the first term of the root, and subtract its cube from the given polynomial.

Take three times the square of the part of the root already found for the trial divisor, divide the first term of the remainder by it, and write the quotient for the next term of the root.

Add together the trial divisor, three times the product of the first term by the last, and the square of the last, for a complete divisor.

Multiply the complete divisor by the last term of the root, and subtract the product from the dividend.

If there are other terms remaining, form a new dividend, and continue the operation in the same manner as before.

The terms of each new dividend must be arranged, if necessary, according to the powers of the leading letter of the root.

EXAMPLES.

2. Find the cube root of $y^6 - 3y^5 + 5y^3 - 3y - 1$.

OPERATION.

$$
\begin{array}{r|l}
y^6 - 3y^5 + 5y^3 - 3y - 1 & y^2 - y - 1 \\
\cline{2-2}
y^6 & \\
\cline{1-1}
3y^4 - 3y^3 + y^2 \;\big|\; -3y^5 + 5y^3 & \\
\quad\quad\quad\quad\quad\quad\;\; -3y^5 + 3y^4 - y^3 & \\
\cline{1-1}
3y^4 - 6y^3 + 3y + 1 \;\big|\; -3y^4 + 6y^3 - 3y - 1 & \\
\quad\quad\quad\quad\quad\quad\;\; -3y^4 + 6y^3 - 3y - 1 & \\
\cline{2-2}
\end{array}
$$

Here, it will be noticed that, in the last complete divisor, $3y^3$ and $-3y^3$ cancel.

3. Extract the cube root of $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

Ans. $a + b + c$.

4. Extract the cube root of $1 - 6y + 12y^2 - 8y^3$.

Ans. $1 - 2y$.

5. Extract the cube root of $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$.

Ans. $x + \frac{1}{x}$.

6. Find the cube root of $\frac{a^6}{b^2} - 6a^4 + 12a^2b^2 - 8b^4$.

Ans. $\frac{a^2}{b^{\frac{2}{3}}} - 2b^{\frac{4}{3}}$.

7. Find the cube root of $x^8 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$. Ans. $x^2 - 2x + 1$.

ANY ROOT OF POLYNOMIALS.

228. In order to establish a general rule for the extraction of roots, it will be necessary to notice the formation of the nth power of a polynomial, n being any entire number whatever. Thus,

$$(a + b)^n = a^n + n\,a^{n-1}b + \ldots\ldots$$

Therefore,

$$\sqrt[n]{a^n + n\,a^{n-1}b + \ldots\ldots} = a + b.$$

The first term of the root, a, is the nth root of a^n, the first term of the power, and the second term of the root, b, may be obtained by dividing the second term of the power by $n\,a^{n-1}$, or by n times the first term of the root, raised to the $(n-1)$th power.

If the root now found be raised to the nth power, and subtracted from the given power, it will be seen that two terms of the required root have been determined.

It will be observed that the process is essentially that of the preceding Articles, simplified by dispensing with completed divisors, and generalized. Hence the

RULE.

Arrange the terms according to the powers of some letter.

Find the required root of the first term for the first term of the root, and subtract its power from the given polynomial.

Divide the first term of the remainder by n *times the* (n—1)*th power of this root, for the second term of the root, and subtract the* n*th power of the root now found from the given quantity.*

If other terms of the root require to be determined, use the same divisor as before, and proceed in like manner till the n*th power of the root becomes equal to the given polynomial.*

This rule is, also, obviously applicable to numbers, by taking n figures in each period.

EXAMPLES.

1. Extract the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

OPERATION.

$$\begin{array}{rl} & x^6 + 6x^5 - 40x^3 + 96x - 64 \;\big|\; x^2 + 2x - 4 \\ (x^2)^3 = & x^6 \\ \hline & 3x^4 \;\big|\; 6x^5 \\ (x^2 + 2x)^3 = & x^6 + 6x^5 + 12x^4 + 8x^3 \\ \hline & 3x^4 \;\big|\; -12x^4 \\ (x^2 + 2x - 4)^3 = & x^6 + 6x^5 - 40x^3 + 96x - 64 \end{array}$$

2. Extract the cube root of $m^6 - 6m^5 + 40m^3 - 96m - 64$.

Ans. $m^2 - 2m - 4$.

3. Extract the square root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$. Ans. $a^2 - ax + x^2$.

4. Required the fifth root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$. Ans. $2x - 1$.

229. When the index of the required root is a multiple of two or more numbers, we may obtain *the root by successive extractions of the simpler roots.*

For, since (Art. 201)

$$(a^n)^m = a^{mn},$$

reversing the involution, we have

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}; \text{ or,}$$

The mn*th root of a quantity is equal to the* m*th root of the* n*th root of that quantity.*

5. Required the fourth root of $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$. Ans. $2x - 3y$.

Here, we may proceed by the rule in the last Article, or find the square root of the square root of the given quantity.

6. Find the sixth root of $a^6 + \frac{1}{a^6} - 6\left(a^4 + \frac{1}{a^4}\right) + 15\left(a^2 + \frac{1}{a^2}\right) - 20$. Ans. $a - \frac{1}{a}$.

RADICALS.

230. A Radical is a root of a quantity indicated either by a radical sign or by a fractional exponent; as, $\sqrt{a}$, $a^{\frac{1}{3}}$, and $2\sqrt[3]{1+a}$.

When the root indicated can be exactly obtained, it is called a *rational quantity*, and when it cannot be exactly obtained, it is called an *irrational* or *surd quantity*.

231. The COEFFICIENT of a radical is the quantity or factor prefixed to it. Thus, in $2\sqrt{2bc^3}$, and $a(c+d)^{\frac{1}{3}}$, 2 and a are the coefficients.

232. The DEGREE of a radical is denoted by the index of the radical sign, or by the denominator of the fractional exponent. Thus,

$\sqrt{a}$, $\sqrt{y}$, $(abc)^{\frac{1}{2}}$, are radicals of the *second* degree;

$\sqrt[3]{x^2}$, $b^{\frac{1}{3}}$, $(2a^2x^3y)^{\frac{1}{3}}$, are radicals of the *third* degree;

$\sqrt[n]{ac}$, $3\sqrt[n]{m}$, $(a+b)^{\frac{1}{n}}$, are radicals of the nth degree.

233. SIMILAR RADICALS are those of the same degree, with the same quantity under the radical sign. Thus, $5\sqrt[3]{ax}$ and $7\sqrt[3]{ax}$ are similar radicals; and also $ay^{\frac{1}{n}}$ and $cy^{\frac{1}{n}}$.

REDUCTION OF RADICALS.

234. REDUCTION OF RADICALS is the process of changing their forms without altering their values.

235. The reduction of radicals depends upon the general principle, that

The root of any quantity is equal to the product of the like roots of its several factors (Art. 212).

CASE I.

236. To reduce radicals to their simplest form.

A radical is in its *simplest form*, when it has under the sign no factor which is a perfect power of the same degree as the radical.

1. Reduce $\sqrt[3]{54a^5x^4}$ to its simplest form.

OPERATION.

$$\sqrt[3]{54\,a^5\,x^4} = \sqrt[3]{27\,a^3\,x^3 \times 2\,a^2\,x}$$
$$= \sqrt[3]{27\,a^3\,x^3}\,\sqrt[3]{2\,a^2\,x}$$
$$= 3\,a\,x\,\sqrt[3]{2\,a^2\,x}$$

We resolve the quantity under the radical sign into two factors, one of which, $\sqrt[3]{27\,a^3\,x^3}$, is the greatest perfect cube contained in the radical quantity, and the other the surd $\sqrt[3]{2\,a^2\,x}$. But $\sqrt[3]{27\,a^3\,x^3} = 3\,a\,x$; consequently, by the general principle (Art. 235), the given radical is equal to the product of $3\,a\,x$ and $\sqrt[3]{2\,a^2\,x}$, or $3\,a\,x\,\sqrt[3]{2\,a^2\,x}$. Hence the

RULE.

Resolve the quantity under the radical sign into two factors, one of which shall contain all the perfect powers of the same degree as the radical. Extract the required root of this factor, multiply the root by the coefficient of the radical, and prefix the product to the other factor, with the radical sign between them.

When the given radical has no coefficient expressed, the coefficient 1 is always understood.

EXAMPLES.

2. Reduce $\sqrt{27\,a^3\,x^5}$ to its simplest form.

Ans. $3\,a\,x^2\,\sqrt{3\,a\,x}$.

3. Reduce $(189\,a^4\,b^3\,c^2)^{\frac{1}{3}}$ to its simplest form.

Ans. $3\,a\,b\,(7\,a\,c^2)^{\frac{1}{3}}$.

4. Reduce $\sqrt{96\,a^2\,x^2}$ to its simplest form.

Ans. $4\,a\,x\,\sqrt{6\,x}$.

5. Reduce $\sqrt[n]{a^{m+n}\,b^n}$ to its simplest form.

Ans. $a\,b\,\sqrt[n]{a^m}$.

6. Reduce $2c\sqrt[4]{768}$ to its simplest form.

Ans. $8c\sqrt[4]{3}$.

7. Reduce $\sqrt{a^3 - a^2b}$ to its simplest form.

Ans. $a\sqrt{a-b}$.

8. Reduce $(ax^2 - 6ax + 9a)^{\frac{1}{2}}$ to its simplest form.

Ans. $(x-3)\,a^{\frac{1}{2}}$.

9. Reduce $\sqrt{(x^2-y^2)\,(x+y)}$ to its simplest form.

Ans. $(x+y)\sqrt{x-y}$.

237. When the given radical is in a fractional form, we may transform it in such a manner as to make the denominator a perfect power of the degree indicated. Then, the surd factor will be an entire quantity.

10. Reduce $\sqrt{\frac{2}{7}}$ to its simplest form.

$$\sqrt{\tfrac{2}{7}} = \sqrt{\tfrac{14}{49}} = \sqrt{\tfrac{1}{49} \times 14} = \tfrac{1}{7}\sqrt{14}.$$

11. Reduce $4\sqrt{\frac{3a^3}{8b}}$ to its simplest form.

$$4\sqrt{\frac{3a^3}{8b}} = 4\sqrt{\frac{3a^3}{8b} \times \frac{2b}{2b}} = 4\sqrt{\frac{6a^3b}{16b^2}} = 4\sqrt{\frac{a^2}{16b^2} \times 6ab}$$

$$= 4 \times \frac{a}{4b}\sqrt{6ab} = \frac{a}{b}\sqrt{6ab}.$$

12. Reduce $\frac{1}{2}\sqrt{\frac{x^3}{15}}$ to its simplest form.

Ans. $\frac{x}{30}\sqrt{15x}$.

13. Reduce $\frac{3}{11}\sqrt{\frac{4}{7}}$ to its simplest form.

Ans. $\frac{6}{77}\sqrt{7}$.

14. Reduce $\sqrt{\frac{a\,b^2}{4\,(a+x)}}$ to its simplest form.

Ans. $\frac{b}{2\,(a+x)}\sqrt{a\,(a+x)}$.

15. Reduce $\frac{a}{a^2-b^2}\sqrt{\frac{a^3\,b-2\,a^2\,b^2+a\,b^3}{b^2}}$ to its simplest form.

Ans. $\frac{a}{b\,(a+b)}\sqrt{a\,b}$.

CASE II.

238. To reduce a rational quantity to the form of a radical.

1. Reduce $2\,x^2$ to the form of the cube root.

OPERATION.

$$2\,x^2 = (2\,x^2)^{\frac{3}{3}} = (8\,x^6)^{\frac{1}{3}} = \sqrt[3]{8\,x^6}.$$

Here, $2\,x^2$ cubed is $8\,x^6$; and, since evolution is the reverse of involution, $8\,x^6$ written under the sign of the root indicated, gives $(8\,x^6)^{\frac{1}{3}}$, or $\sqrt[3]{8\,x^6}$, as the required form. Hence the

RULE.

Raise the quantity to the power indicated by the given root, and write it under the corresponding radical sign.

EXAMPLES.

2. Reduce $a+x$ to a radical of the nth degree.

Ans. $\sqrt[n]{(a+x)^n}$.

3. Reduce $x-y^2$ to the form of the square root.

Ans. $\sqrt{(x-y^2)^2}$.

4. Reduce $\frac{x^2}{3}$ to the form of the cube root.

Ans. $\sqrt[3]{\frac{x^6}{27}}$.

5. Reduce $\frac{x^3}{x-y}$ to the form of the fourth root.

Ans. $\sqrt[4]{\frac{x^8}{(x-y)^4}}$.

239. A *coefficient,* or a *factor of a coefficient,* of a radical may be placed under the radical sign, by *raising it to the power indicated by the radical, and multiplying the quantity already under the sign by the result.*

6. Introduce the coefficient of $2xy\sqrt{5}$ under the radical sign.

$$2xy\sqrt{5} = \sqrt{(2xy)^2 \times 5} = \sqrt{20x^2y^2}.$$

7. Reduce $x(2a - x)^{\frac{1}{2}}$ to a radical without a coefficient. Ans. $(2ax^2 - x^3)^{\frac{1}{2}}$.

8. Introduce the factor 3 of the coefficient of $3a\sqrt[n]{m}$ under the radical sign. Ans. $a\sqrt[n]{3^n m}$.

9. Reduce $\frac{1}{2}\sqrt[4]{b^2}$ to a radical without a coefficient.

Ans. $\sqrt[4]{\frac{b^2}{16}}$.

10. Reduce $\frac{2x}{3}\sqrt[3]{\frac{9}{4x^3}}$ to a simple radical form.

Ans. $\sqrt[3]{\frac{2x}{3}}$.

11. Introduce the coefficient of $\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}$ under the radical sign. Ans. $\sqrt{\frac{x+1}{x-1}}$.

CASE III.

240. To reduce radicals of different degrees to equivalent ones having a common index.

1. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ to radicals having a common index.

OPERATION.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}} = (a^3)^{\frac{1}{6}} = \sqrt[6]{a^3},$$
$$b^{\frac{1}{3}} = b^{\frac{2}{6}} = (b^2)^{\frac{1}{6}} = \sqrt[6]{b^2}.$$

From Article 238, it is evident that $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ are equal to $a^{\frac{3}{6}}$ and $b^{\frac{2}{6}}$, respectively. That is, we may reduce the given exponents to equivalent ones, having a common denominator. But $a^{\frac{3}{6}}$ is equal to $(a^3)^{\frac{1}{6}}$, or $\sqrt[6]{a^6}$, and $b^{\frac{2}{6}}$ is equal to $(b^2)^{\frac{1}{6}}$, or $\sqrt[6]{b^2}$. Hence the

RULE.

Reduce the given exponents to a common denominator; raise each quantity to the power denoted by the numerator of the reduced exponent, and indicate the root denoted by the denominator.

A common index may also be found, without the use of fractional exponents, by multiplying the index and the exponents of each of the given radicals by such a quantity as will make its index the least common multiple of the given indices.

EXAMPLES.

2. Reduce a^2 and $a^{\frac{1}{2}}$ to radicals having the common index 4. Ans. $\sqrt[4]{a^8}$ and $\sqrt[4]{a^2}$.

3. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to radicals having a common index. Ans. $\sqrt[4]{a^2}$ and $\sqrt[4]{b}$.

4. Reduce $x^{\frac{1}{m}}$ and $y^{\frac{1}{n}}$ to radicals having a common index. Ans. $\sqrt[mn]{x^n}$ and $\sqrt[mn]{y^m}$.

5. Reduce $\sqrt[4]{a}$, $\sqrt[6]{5\,b}$, and $\sqrt[8]{3\,c}$ to radicals having a common index. Ans. $\sqrt[24]{a^6}$, $\sqrt[24]{625\,b^4}$, and $\sqrt[24]{27\,c^3}$.

6. Reduce $\sqrt[3]{a+c}$ and $\sqrt{a-c}$ to radicals having a common index. Ans. $\sqrt[6]{(a+c)^2}$ and $\sqrt[6]{(a-c)^3}$.

ADDITION OF RADICALS.

241. When radicals to be added are similar, the common radical part, with the sum of their coefficients, will constitute the sum of the radicals.

But if they are dissimilar, it is obvious that their addition can only be indicated (Art. 53).

Hence the following

RULE.

Reduce each radical, if necessary, to its simplest form. If, then, the radicals are similar, add their coefficients, and to the sum annex the common radical; but if they are dissimilar, indicate the addition by the proper sign.

EXAMPLES.

1. Find the sum of $\sqrt{3\,a^2\,b}$ and $\sqrt{3\,x^2\,b}$.

OPERATION.

$$\begin{aligned}\sqrt{3\,a^2\,b} &= \sqrt{3\,b\times a^2} = a\sqrt{3\,b}\\ \sqrt{3\,x^2\,b} &= \sqrt{3\,b\times x^2} = x\sqrt{3\,b}\\ \text{Sum} &= (a+x)\sqrt{3\,b}\end{aligned}$$

2. Find the sum of $5\sqrt{a\,b}$ and $3\sqrt{a\,b}$.

Ans. $8\sqrt{a\,b}$.

3. Find the sum of $\sqrt[3]{375}$ and $\sqrt[3]{192}$.

Ans. $9\sqrt[3]{3}$.

4. Find the sum of $3\sqrt{a^2 b}$ and $5\sqrt{16 a^4 b}$.

Ans. $(3a + 20a^2)\sqrt{b}$.

5. Find the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

Ans. $\frac{4}{5}\sqrt{10}$.

6. Find the sum of $(b^4 y)^{\frac{1}{3}}$ and $(b y^4)^{\frac{1}{3}}$.

Ans. $(b + y)(by)^{\frac{1}{3}}$.

7. Add $8\sqrt{\frac{3}{4}}$, $-\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$, and $-2\sqrt{\frac{3}{16}}$.

Ans. $14\frac{1}{2}\sqrt{3}$.

8. Add $\sqrt{48 a b^2}$, $\sqrt{75 a^3}$, and $\sqrt{3a(a - 9b)^2}$.

Ans. $(6a - 5b)\sqrt{3a}$.

9. Required the sum of $\sqrt[3]{\frac{27 a^5 d}{2b}}$ and $-\sqrt[3]{\frac{a^2 d}{2b}}$.

Ans. $(3a - 1)\sqrt[3]{\frac{a^2 d}{2b}}$.

10. Required the sum of $3(242 a^5 b^5)^{\frac{1}{2}}$ and $11(2a^3 b^3)^{\frac{1}{2}}$.

Ans. $11(3a^2 b^2 + ab)(2ab)^{\frac{1}{2}}$.

11. Required the value of $\sqrt{24} + 2\sqrt{72} + a\sqrt{b x^2}$.

Ans. $2\sqrt{6} + 12\sqrt{2} + ax\sqrt{b}$.

12. Required the value of $\sqrt[3]{125 b^4 - 250 a b^3} + \sqrt[3]{64 a^3 b - 128 a^4}$. Ans. $(5b + 4a)\sqrt[3]{b - 2a}$.

SUBTRACTION OF RADICALS.

242. When two radicals are similar, the common radical part, with the difference of their coefficients, will constitute their difference. But, while they are dissimilar, their difference can only be indicated.

Hence the following

RULE.

Reduce each radical, if necessary, to its simplest form. If, then, the radicals are similar, find the difference of the coefficients, and to the result annex the common radical; but if they are dissimilar, indicate the subtraction by the proper sign.

EXAMPLES.

1. Find the difference between $\sqrt{108\,a\,x^2}$ and $\sqrt{48\,a\,x^2}$.

OPERATION.

$$\sqrt{108\,a\,x^2} = \sqrt{36\,x^2 \times 3\,a} = 6\,x\sqrt{3\,a}$$
$$\sqrt{48\,a\,x^2} = \sqrt{16\,x^2 \times 3\,a} = 4\,x\sqrt{3\,a}$$
$$\text{Difference} = 2\,x\sqrt{3\,a}$$

2. Find the difference between $2\sqrt[3]{320}$ and $3\sqrt[3]{40}$.

Ans. $2\sqrt[3]{5}$.

3. From $\sqrt{4\,a\,x^2}$ take $3\,x\sqrt{9\,a}$. Ans. $-7\,x\sqrt{a}$.

4. Find the difference between $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{2}{3}\frac{5}{5}}$.

Ans. $\frac{2}{15}\sqrt[3]{75}$.

5. Find the difference between $\frac{3}{7}\sqrt[3]{a^3\,b}$ and $\frac{2}{5}\sqrt[3]{a^6\,b}$.

Ans. $\left(\frac{3\,a}{7} - \frac{2\,a^2}{5}\right)\sqrt[3]{b}$.

6. Simplify $4\sqrt{1+x^2} - 4\sqrt[3]{x^2+1}$.

Ans. $4\,(\sqrt{x^2+1} - \sqrt[3]{x^2+1})$.

7. Simplify $3\,b\,(2\,a^5\,b^2)^{\frac{1}{3}} - 7\,(2\,a^5\,b^5)^{\frac{1}{3}} + 8\,a\,(2\,a^2\,b^5)^{\frac{1}{3}}$.

Ans. $4\,a\,b\,(2\,a^2\,b^2)^{\frac{1}{3}}$.

8. Required the difference between $m\sqrt{1+\left(\frac{n}{m}\right)^{\frac{1}{2}}}$ and $n\sqrt{1+\left(\frac{m}{n}\right)^{\frac{1}{2}}}$. Ans. $\left(m^{\frac{3}{2}} - n^{\frac{3}{2}}\right)\sqrt{m^{\frac{1}{2}} + n^{\frac{1}{2}}}$.

MULTIPLICATION OF RADICALS.

243. The multiplication of radicals depends upon the principle, that

The product of like roots of two quantities is equal to the same root of their product (Art. 212).

That is, let a and b be any two quantities; then we have

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}, \quad \text{or} \quad (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}},$$

and, conversely,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}, \quad \text{or} \quad a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}.$$

Therefore, if $c\sqrt[n]{a}$ and $d\sqrt[n]{b}$ be two radicals with coefficients and a common index; then (Art. 63),

$$c\sqrt[n]{a} \times d\sqrt[n]{b} = c \times d \times \sqrt[n]{a} \times \sqrt[n]{b} = cd\sqrt[n]{ab}.$$

Hence the

RULE.

Reduce the radical parts, if necessary, to those having a common index; then multiply the coefficients together for the coefficient of the product, and the parts under the radicals for the radical part.

EXAMPLES.

1. Multiply $3\sqrt{2a}$ by $2\sqrt[3]{3a}$.

OPERATION.

$$\begin{aligned} 3\sqrt{2a} \times 2\sqrt[3]{3a} &= 3\sqrt[6]{2^3a^3} \times 2\sqrt[6]{3^2a^2} \\ &= 3 \times 2\sqrt[6]{2^3a^3 \times 3^2a^2} \\ &= 6\sqrt[6]{72a^5} \end{aligned}$$

2. Multiply $\sqrt[m]{a}$ by $\sqrt[n]{b}$. Ans. $\sqrt[mn]{a^n b^m}$.

3. Multiply $5\,a^{\frac{1}{2}}$ by $3\,a^{\frac{1}{3}}$. Ans. $15\sqrt[6]{a^5}$.

4. Multiply $4\sqrt{12}$ by $3\sqrt{2}$. Ans. $24\sqrt{6}$.

5. Multiply $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ by $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$. Ans. $\frac{1}{8}\sqrt[3]{15}$.

6. Multiply $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$.

Ans. $\sqrt[12]{(a+b)^{13}}$.

7. Multiply $b^{-2}\sqrt[4]{a^{-3}}$ by $a^{\frac{5}{6}}b^{\frac{1}{2}}c$.

Ans. $\frac{c}{b^2}\sqrt[12]{a\,b^6}$.

8. Multiply $4\,a\sqrt{\frac{3\,a^2}{8\,b}}$ by $\frac{1}{2}\,a\sqrt{\frac{3}{2\,b}}$. Ans. $\frac{3\,a^3}{2\,b}$.

244. When either or both of the radicals are connected with other quantities by the sign + or −, *each term of the multiplicand must be multiplied by each term of the multiplier* (Art. 68).

9. Multiply $a+2\sqrt{b}$ by $a-\sqrt{b}$.

FIRST OPERATION.

$$\begin{array}{l} a+2\sqrt{b} \\ a-\sqrt{b} \\ \hline a^2+2\,a\sqrt{b} \\ \quad -a\sqrt{b}-2\sqrt{b^2} \\ \hline a^2+a\sqrt{b}-2\,b \end{array}$$

SECOND OPERATION.

$$\begin{array}{l} a+2\,b^{\frac{1}{2}} \\ a-b^{\frac{1}{2}} \\ \hline a^2+2\,a\,b^{\frac{1}{2}} \\ \quad -a\,b^{\frac{1}{2}}-2\,b^{\frac{2}{2}} \\ \hline a^2-a\,b^{\frac{1}{2}}-2\,b \end{array}$$

10. Multiply $x-\sqrt{xy}+y$ by $\sqrt{x}+\sqrt{y}$.

Ans. $x\sqrt{x}+y\sqrt{y}$.

11. Multiply $\sqrt{a}+c\sqrt[3]{b}$ by $\sqrt{a}-c\sqrt[3]{b}$.

Ans. $a-c^2\sqrt[3]{b^2}$.

12. Multiply $\sqrt{a}+\sqrt{b}+\sqrt{c}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

Ans. $a+b-c+2\sqrt{ab}$.

DIVISION OF RADICALS.

245. Division of radicals depends upon the principle that

The quotient of like roots of two quantities is equal to the same root of their quotient.

That is, let a and b represent two quantities; then,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}, \quad \text{or} \quad \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

Therefore, if $cd\sqrt[n]{ab}$ and $d\sqrt[n]{b}$ be two radicals with coefficients and a common index; then

$$\frac{cd\sqrt[n]{ab}}{d\sqrt[n]{b}}=\frac{cd}{d}\sqrt[n]{\frac{ab}{b}}=c\sqrt[n]{a}.$$

Hence the

RULE.

Reduce the radical parts, if necessary, to those having a common index; then divide the coefficient of the dividend by the coefficient of the divisor, and the radical part of the dividend by that of the divisor, and prefix the first quotient to the last, written under the common index.

EXAMPLES.

1. Divide $16a^2\sqrt[3]{b^2}$ by $8a\sqrt{b}$.

OPERATION.

$$\frac{16a^2\sqrt[3]{b^2}}{8a\sqrt{b}}=\frac{16a^2\sqrt[6]{b^4}}{8a\sqrt[6]{b^3}}=\frac{16a^2}{8a}\sqrt[6]{\frac{b^4}{b^3}}=2a\sqrt[6]{b}$$

2. Divide $6\sqrt{96}$ by $3\sqrt{8}$. Ans. $4\sqrt{3}$.

3. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$. Ans. $2\sqrt[3]{4}$.

4. Divide $\sqrt{96\,a^2}$ by $4\,a$. Ans. $\sqrt{6}$.

5. Divide $(x^2y^3)^{\frac{1}{4}}$ by $(x^2y^3)^{\frac{1}{2}}$. Ans. $(x^2y^3)^{-\frac{1}{4}}$.

6. Divide $2\sqrt[3]{b\,c}$ by $3\sqrt{a\,c}$.

Ans. $\frac{2}{3\,a\,c}\sqrt[6]{a^3\,b^2\,c^5}$.

7. Divide $b\,c\,(a^3x^2)^{\frac{1}{n}}$ by $c\,(a\,x)^{\frac{1}{n}}$. Ans. $b\,(a^2x)^{\frac{1}{n}}$.

8. Divide $\sqrt{a\,x^3}$ by $\sqrt{b\,x}$. Ans. $\frac{x}{b}\sqrt{a\,b}$.

9. Divide $\sqrt[3]{\frac{a\,x}{2\,b\,y^2}}$ by $\sqrt[6]{\frac{2\,a^2\,x^3}{3\,b\,y^3}}$. Ans. $\sqrt[6]{\frac{3}{8\,b\,y}}$.

10. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$. Ans. 2.

11. Divide $a+b-c+2\sqrt{a\,b}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

Ans. $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

12. Divide $\sqrt[3]{a^5\,b^2+a^3\,b^4}$ by $\sqrt[3]{\frac{a^2-b^2}{8\,b}}$.

Ans. $2\,a\,b\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}$.

POWERS OF RADICALS.

246. Involution of radicals depends upon the same general principles as involution of rational quantities.

That is, let a represent any quantity; then, we have the nth power of $\sqrt[m]{a}$, or

$$(\sqrt[m]{a})^n = \sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}\ \ldots\ldots \text{ to } n \text{ factors} = \sqrt[m]{a^n},$$

or,

$$\left(a^{\frac{1}{m}}\right)^n = a^{\frac{1}{m}}\times a^{\frac{1}{m}}\times a^{\frac{1}{m}}\ \ldots\ldots \text{ to } n \text{ factors} = a^{\frac{n}{m}}.$$

Therefore, if $b\sqrt[m]{a}$ be a radical with a coefficient, then (Art. 200),

$$(b\sqrt[m]{a})^n = b^n \times \sqrt[m]{a^n} = b^n\sqrt[m]{a^n}.$$

Hence the

RULE.

Raise the coefficient, if any, to the required power, and write the radical part, raised to the same power, under the given sign.

If a quantity have fractional exponents, the involution may be performed by multiplying them by the exponent of the required power (Art. 203).

If the radical has quantities connected by + or —, perform the involution by multiplication of the several terms.

EXAMPLES.

1. Raise $\frac{2}{3}\sqrt{3}$ to the third power.

OPERATION.

$$(\tfrac{2}{3}\sqrt{3})^3 = (\tfrac{2}{3})^3 \times \sqrt{3^3} = \tfrac{8}{27}\sqrt{27}$$
$$= \tfrac{8}{27}\sqrt{9 \times 3} = \tfrac{8}{9}\sqrt{3} \quad \text{Ans.}$$

2. Raise $9x\sqrt[3]{3ab^2}$ to the second power.

Ans. $81bx^2\sqrt[3]{9a^2b}$.

3. Raise $\sqrt[4]{ab^2}$ to the second power. Ans. $b\sqrt{a}$.

4. Raise $\frac{1}{3}\sqrt{3}$ to the third power. Ans. $\frac{1}{9}\sqrt{3}$.

5. Raise $-\sqrt[3]{a^2}$ to the fourth power. Ans. $a^2\sqrt[3]{a^2}$.

6. Find the square of $2+\sqrt{3}$. Ans. $7+4\sqrt{3}$.

7. Find the cube of $a - bx^{\frac{1}{2}}$.

Ans. $a^3 - 3a^2bx^{\frac{1}{2}} + 3ab^2x - b^3x^{\frac{3}{2}}$.

8 Find the cube of $2\sqrt{a-b}$.

Ans. $8(a-b)\sqrt{a-b}$.

ROOTS OF RADICALS.

247. Evolution of radicals depends upon the same general principles as evolution of rational quantities.

That is, let a represent any quantity; then, by Art. 229, we have for the mth root of the nth root of a,

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}, \quad \text{or} \quad \sqrt[m]{a^{\frac{1}{n}}} = a^{\frac{1}{mn}}.$$

Let it now be required to find a root of a radical with a coefficient, as the cube root of $8\sqrt{a^3}$, or the square root of $20\sqrt{5a}$; then, by Art. 212, we have

$$\sqrt[3]{8\sqrt{a^3}} = \sqrt[3]{8} \times \sqrt[3]{\sqrt{a^3}} = 2\sqrt{a}.$$

Also, by Art. 236,

$$\sqrt{20\sqrt{5a}} = \sqrt{4 \times 5\sqrt{5a}};$$

and $$\sqrt{4} \times \sqrt{5\sqrt{5a}} = 2\sqrt{\sqrt{125a}} = 2\sqrt[4]{125a}.$$

Hence the

RULE.

Extract the required root of the coefficient, if any, when a complete power of the required degree, otherwise introduce it under the radical sign.

Extract the required root of the quantity under the radical sign, when a complete power of the required degree, otherwise multiply the index of the radical by the index of the required root.

When only a factor of the coefficient is a complete power of the same degree as the root required, take the root of that factor, and introduce the other factor under the sign.

When a quantity is affected by a fractional exponent, its evolution may be performed by dividing this exponent by the index of the required root (Art. 223).

EXAMPLES.

1. Find the cube root of the square root of a.

Ans. $\sqrt[6]{a}$.

2. Find the cube root of $64\sqrt[3]{7}$. Ans. $4\sqrt[9]{7}$.

3. Find the square root of $a^2\sqrt[3]{5}$. Ans. $a\sqrt[6]{5}$.

4. Find the square root of $24\sqrt{3a}$.

Ans. $2\sqrt[4]{108a}$.

5. Find the cube root of $(x+y)\sqrt{x+y}$.

Ans. $\sqrt{x+y}$.

6. Find the cube root of $\frac{a}{5}\sqrt{\frac{a}{5}}$. Ans. $\frac{1}{5}\sqrt{5a}$.

7. Find the fourth root of $2\sqrt{a^2-x}$.

Ans. $\sqrt[6]{4(a^2-x)}$.

8. Find the mth root of $5(a+b)^{\frac{1}{2}}$.

Ans. $5^{\frac{1}{m}}(a+b)^{\frac{1}{2m}}$.

9. Find the square root of $9a+36(3ax)^{\frac{1}{2}}+108x$.

Ans. $3a^{\frac{1}{2}}+6(3x)^{\frac{1}{2}}$.

RATIONALIZATION.

248. RATIONALIZATION is the process of removing the radical sign from a quantity by multiplication.

It is often useful to transform a fraction having an irrational denominator, into one whose denominator is rational. This is especially advantageous when the numerical value of the quantities is required.

249. *A factor may be found which will rationalize any monomial.*

The quantity $\sqrt{a}$, or $a^{\frac{1}{2}}$, is made rational by multiplying it by $\sqrt{a}$, or $a^{\frac{1}{2}}$.

For, $\qquad \sqrt{a} \times \sqrt{a} = a, \text{ or } a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a.$

Also, $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, is made rational by multiplying it by $a^{\frac{2}{3}}$.

For, $\qquad a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a.$

Likewise, $a^{\frac{4}{5}}$ is made rational by being multiplied by $a^{\frac{1}{5}}$.

For, $\qquad a^{\frac{4}{5}} \times a^{\frac{1}{5}} = a.$

That is,

The required factor is the same as the given quantity, but with such a fractional exponent as, when added to the fractional exponent of that quantity, shall be equal to an integer.

250. *A factor may be found which will rationalize any binomial.*

Suppose the binomial to be rationalized to be in the form of $\sqrt{a} \pm \sqrt{b}$.

Then, since the product of the sum and difference of two quantities is equal to the difference of their squares (Art. 90), we have

$$(\sqrt{a} + \sqrt{b})\ (\sqrt{a} - \sqrt{b}) = a - b,$$

which is rational. That is, when the binomial is of the second degree,

The required factor is the given binomial with one of the signs changed.

This form occurs most frequently in practice; but the following is a more general demonstration: —

1. Suppose the binomial $\sqrt[p]{a} + \sqrt[q]{b}$, or $a^{\frac{1}{p}} + b^{\frac{1}{q}}$. Put $x = a^{\frac{1}{p}}$, $y = b^{\frac{1}{q}}$; let n be the least common multiple of p and q; then x^n and y^n are both rational. Now (Art. 97, 98),

$$(x+y)\,(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots\ldots \pm y^{n-1}) = x^n \pm y^n,$$

where the upper or lower sign must be taken according as n is odd or even. Hence,

$$x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots\ldots \pm y^{n-1}$$

is a factor which will rationalize $x + y$.

2. Suppose the binomial $\sqrt[p]{a} - \sqrt[q]{b}$, or $a^{\frac{1}{p}} - b^{\frac{1}{q}}$. Take x, y, and n as before. Now (Art. 96),

$$(x-y)\,(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + y^{n-1}) = x^n - y^n.$$

Hence,

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + y^{n-1}$$

is a factor which will rationalize $x - y$.

Let it be required to rationalize $\sqrt{a} + \sqrt[3]{b}$, or $a^{\frac{1}{2}} + b^{\frac{1}{3}}$. Here, since $n = 6$, an even number, we have from (1),

$$\frac{x^6 - y^6}{x+y} = \frac{a^{\frac{6}{2}} - b^{\frac{6}{3}}}{a^{\frac{1}{2}} + b^{\frac{1}{3}}} = a^{\frac{5}{2}} - a^{\frac{4}{2}}b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - a^{\frac{2}{2}}b^{\frac{3}{3}} + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}}$$

$$= a^{\frac{5}{2}} - a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} - a\,b + a^{\frac{1}{2}} b^{\frac{4}{3}} - b^{\frac{5}{3}},$$

the rationalizing factor; and

$$(a^{\frac{1}{2}} + b^{\frac{1}{3}})\,(a^{\frac{5}{2}} - a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} - a\,b + a^{\frac{1}{2}} b^{\frac{4}{3}} - b^{\frac{5}{3}})$$

$$= a^{\frac{6}{2}} - b^{\frac{6}{3}} = a^3 - b^2,$$

the rationalized result.

251. A *trinomial* in the form of $\sqrt{a} + \sqrt{b} + \sqrt{c}$ may be transformed into a binomial form by multiplying it by

itself with the sign of one of the terms changed, and then rationalized. Thus,

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\ (\sqrt{a}+\sqrt{b}-\sqrt{c}) = a+b-c+2\sqrt{ab},$$

$$(a+b-c+2\sqrt{ab})\ (a+b-c-2\sqrt{ab}) = (a+b-c)^2-4ab,$$

or $$a^2+b^2+c^2-2ab-2bc-2ac,$$

a rational result.

EXAMPLES.

1. Find a factor which will rationalize $3\sqrt{a^3}$.

Ans. $\sqrt{a}$.

2. Find a factor which will rationalize $a+\sqrt{b}$.

Ans. $a-\sqrt{b}$.

3. Rationalize $\sqrt[4]{5}+\sqrt[4]{3}$. Ans. $5-3$, or 2.

4. Find a factor which will rationalize $\sqrt[5]{(a-b)^2}$.

Ans. $\sqrt[5]{(a-b)^3}$.

5. Find a factor which will rationalize $a^{\frac{1}{2}}-b^{\frac{2}{3}}$.

Ans. $a^{\frac{5}{2}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}+ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}+b^{\frac{10}{3}}$.

6. Reduce $\frac{b}{\sqrt{a}}$ to a fraction whose denominator shall be rational.

$$\frac{b}{\sqrt{a}}\times\frac{\sqrt{a}}{\sqrt{a}}=\frac{b\sqrt{a}}{a},\ \text{Ans.}$$

7. Reduce $\frac{b}{\sqrt[3]{a}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{b\sqrt[3]{a^2}}{a}$.

8. Reduce $\frac{b}{\sqrt[n]{a}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{b\sqrt[n]{a^{n-1}}}{a}$.

9. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to its simplest form.

Ans. $\sqrt{5}+\sqrt{2}$.

10. What factor will simplify $\frac{a}{\sqrt{b}+\sqrt{c}}$?

Ans. $\frac{\sqrt{b}-\sqrt{c}}{\sqrt{b}-\sqrt{c}}$.

11. Rationalize $\sqrt{8}-\sqrt{1}-\sqrt{3}$. Ans. 4.

12. Reduce $\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$ to a fraction whose denominator shall be rational. Ans. $\frac{a-\sqrt{a^2-x^2}}{x}$.

13. Find the approximate value of $\frac{1}{\sqrt{3}+1}$ to three places of decimals.

$$\frac{1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}=\frac{\sqrt{3}-1}{3-1}=\frac{\sqrt{3}-1}{2}$$

$$=\frac{1.732-1}{2}=.366, \text{ Ans.}$$

14. Find the approximate value of $\frac{\sqrt{2}}{3-\sqrt{2}}$ to four places of decimals. Ans. .8918.

IMAGINARY QUANTITIES.

252. An Imaginary Quantity is an indicated even root of a negative quantity. Thus,

$\sqrt{-4}$ and $\sqrt[4]{-a^2}$ are imaginary quantities.

In contradistinction, all other quantities, rational and irrational, are called *real* quantities.

253. When an expression consists of a real part joined with an imaginary part, the whole is spoken of as imaginary, on account of the presence of the latter. Thus,

$a + b\sqrt{-1}$, considered as a whole, is imaginary.

254. Let a denote any real quantity; then the square roots of $-a^2$ will be expressed by

$$\pm\sqrt{-a^2}.$$

Now, $-a^2$ may be considered as the product of a^2 and -1; so if we suppose that the square roots of this product can be formed, in the same manner as if both factors were positive, by multiplying together the square roots of the factors, we have

$$\pm\sqrt{-a^2} = \pm\sqrt{a^2}\times\sqrt{-1} = \pm a\sqrt{-1},$$

in which $\sqrt{-1}$ is the *imaginary factor*, and a its coefficient. Thus, we may regard $\sqrt{-1}$ as a universal factor of every imaginary quantity, and use it in our investigations as the only symbol of such a quantity.

255. Imaginary quantities may be added, subtracted, and divided the same as other radicals, but with regard to multiplication, the common rule requires some modification.

If we take the product of two imaginary quantities in which the imaginary parts are *equal*, it is evident that the sign of the product is changed by removing the radical. Thus,

$$b\sqrt{-a}\times c\sqrt{-a} = bc(-a) = -abc.$$

But if we take two unequal imaginary quantities, $\sqrt{-a}$ and $\sqrt{-b}$, by the common rule (Art. 243) we have

$$(\sqrt{-a})\,(\sqrt{-b}) = \sqrt{ab}.$$

Now, since the quantity whose root is to be extracted was not produced by that root, but from two *unequal* fac-

tors, it does not immediately appear whether the result obtained is to be taken *positively* or *negatively*. We may, however, resolve the imaginary quantities into two factors, of which one is $\sqrt{-1}$ (Art. 254). Then we have

$$\begin{aligned}(+\sqrt{-a})\,(+\sqrt{-b}) &= (+\sqrt{a}\times\sqrt{-1})\,(+\sqrt{b}\times\sqrt{-1})\\ &= +\sqrt{ab}\times(\sqrt{-1})^2\\ &= +\sqrt{ab}\times(-1)\\ &= -\sqrt{ab}.\end{aligned}$$

Hence it appears that the result is properly negative.

In like manner, it may be shown, that

$$(-\sqrt{-a})\,(-\sqrt{-b}) = -\sqrt{ab},$$

and $$(+\sqrt{-a})\,(-\sqrt{-b}) = +\sqrt{ab}.$$

Whence, also, it appears that like signs produce —, and unlike signs +. That is,

The product of two imaginary terms is real, with the sign before the radical as by the common rule reversed.

256. Formulas for the powers of $\sqrt{-1}$ are obtained immediately by multiplication. Thus,

$$(\sqrt{-1})^1 = \sqrt{-1}; \quad (\sqrt{-1})^{4n+1} = \sqrt{-1}; \quad (1)$$

$$(\sqrt{-1})^2 = -1; \quad (\sqrt{-1})^{4n+2} = -1; \quad (2)$$

$$(\sqrt{-1})^3 = -\sqrt{-1}; \quad (\sqrt{-1})^{4n+3} = -\sqrt{-1}; \quad (3)$$

$$(\sqrt{-1})^4 = +1; \quad (\sqrt{-1})^{4n} = +1; \quad (4)$$

and so on, in which n may be 0, or any positive whole number; so that all the powers of $\sqrt{-1}$ form a regular successive repetition of only four different terms, two *real*, ± 1, and two *imaginary*, $\pm\sqrt{-1}$. Hence,

To obtain any power of $\sqrt{-1}$, divide the exponent of the required power by 4, and the power of $\sqrt{-1}$ indicated by the remainder will be equal to the one required.

EXAMPLES.

1. Multiply $4\sqrt{-3}$ by $2\sqrt{-2}$. Ans. $-8\sqrt{6}$.

2. Multiply $-3\sqrt{-a}$ by $4\sqrt{-b}$. Ans. $12\sqrt{ab}$.

3. Find the sum of $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$.
Ans. $2a$.

4. Required the fifth power of $\sqrt{-a^2}$.
Ans. $a^5\sqrt{-1}$.

5. Multiply $a+\sqrt{-b}$ by $a-\sqrt{-b}$.
Ans. a^2+b.

6. Divide $\sqrt{-a}$ by $\sqrt{-b}$.
Ans. $\sqrt{\frac{a}{b}}$.

7. Multiply $4+\sqrt{-7}$ by $\sqrt{-2}$.
Ans. $4\sqrt{-2}-\sqrt{14}$.

8. Multiply $3+4\sqrt{-1}$ by $3+4\sqrt{-1}$.
Ans. $-7+24\sqrt{-1}$.

9. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$ (Art. 250).
Ans. $\sqrt{-1}$.

10. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$.
Ans. $1+\sqrt{-2}$.

11. Find the cube of $c\pm\sqrt{-d}$.
Ans. $c^3\pm 3c^2\sqrt{-d}-3cd\pm d\sqrt{-d}$.

12. Simplify $\sqrt{-a^2}\times\sqrt{-b^2}\times\sqrt{-c^2}$.
Ans. $abc\,(\sqrt{-1})^3=-abc\sqrt{-1}$.

13. Simplify $(\sqrt{-a^2})\,(\sqrt{-b^2})\,(\sqrt{-c^2})\,(\sqrt{-d^2})$.
Ans. $abcd\,(\sqrt{-1})^4=abcd$.

14. Simplify $\frac{a+\sqrt{-b}}{a-\sqrt{-b}}+\frac{a-\sqrt{-b}}{a+\sqrt{-b}}$.

Ans. $\frac{2(a^2-b)}{a^2+b}$.

PROPERTIES OF QUADRATIC SURDS.

257. A Quadratic Surd is the indicated square root of an imperfect square.

258. A Binomial Surd is a binomial in which one or both of the terms are irrational.

259. *The product of two dissimilar quadratic surds, in their simplest form, is irrational.*

Let $\sqrt{a}$ and $\sqrt{b}$ be two quadratic surds in their simplest form (Art. 236); and, if possible, suppose

$$\sqrt{a}\,\sqrt{b} = ac,$$

where ac is rational. Squaring both members, we have

$$ab = a^2c^2,$$

or,
$$b = ac^2;$$

whence,
$$\sqrt{b} = c\sqrt{a};$$

that is, $\sqrt{a}$ and $\sqrt{b}$ have the same irrational part, and are therefore similar (Art. 233), which is contrary to the supposition.

260. *The sum or difference of two dissimilar quadratic surds cannot be equal to a rational quantity.*

Let $\sqrt{a}$ and $\sqrt{b}$ be the two surds; and, if possible, suppose

$$\sqrt{a} \pm \sqrt{b} = c,$$

where c is rational. Transposing, and squaring both members, we have

$$a = c^2 \mp 2c\sqrt{b} + b;$$

whence $$\pm \sqrt{b} = \frac{b + c^2 - a}{2c};$$

that is, an irrational quantity is equal to a rational quantity, which is absurd.

261. *One quadratic surd cannot be composed of two dissimilar quadratic surds.*

If possible, suppose

$$\sqrt{c} = \sqrt{a} \pm \sqrt{b};$$

then, by squaring, we have

$$c = a \pm 2\sqrt{ab} + b;$$

whence $$\pm 2\sqrt{ab} = c - a - b,$$

or a surd equal to a rational quantity, which is impossible.

262. *In any equation which involves rational quantities and quadratic surds, the rational parts on the opposite sides are equal, and also the irrational parts.*

Let $$x \pm \sqrt{y} = a \pm \sqrt{b}$$

be the equation, in which, if x is not equal to a, suppose $x = a \pm m$.

Then, $$a \pm m \pm \sqrt{y} = a \pm \sqrt{b};$$

whence, $$m = \sqrt{b} - \sqrt{y}.$$

That is, a rational quantity is equal to the difference of two quadratic surds, which, by Art. 260, is impossible. Hence, $x = a$, and consequently $\sqrt{y} = \sqrt{b}$.

263. *If* $\sqrt{a + \sqrt{b}}$ *is equal to* $x + \sqrt{y}$, *then will* $\sqrt{a - \sqrt{b}}$ *be equal to* $x - \sqrt{y}$.

For, since

$$\sqrt{a + \sqrt{b}} = x + \sqrt{y},$$

by squaring, we have

$$a + \sqrt{b} = x^2 + 2x\sqrt{y} + y. \quad (1)$$

By Art. 262, $$a = x^2 + y, \quad (2)$$

and $$\sqrt{b} = 2x\sqrt{y}. \quad (3)$$

Subtracting (3) from (2), we have

$$a - \sqrt{b} = x^2 - 2x\sqrt{y} + y, \quad (4)$$

whence, $$\sqrt{a - \sqrt{b}} = x - \sqrt{y}. \quad (5)$$

In like manner, it may be shown, if

$$\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y},$$

then $$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

264. In the solution of certain equations, it frequently becomes necessary to extract the square root of binomial surds in the form of

$$a \pm \sqrt{b}.$$

265. To obtain a rule for the operation, when possible, assume

$$\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}, \quad (1)$$

where one or both terms of the second member are irrational. But by Art. 263,

$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}. \quad (2)$$

Multiplying (1) and (2) together, we have

$$\sqrt{a^2 - b} = x - y. \quad (3)$$

Squaring equation (1), we obtain

$$a + \sqrt{b} = x + 2\sqrt{xy} + y;$$

whence, by Art. 262, $$a = x + y. \quad (4)$$

Adding and subtracting (3) and (4), and factoring,

$$x = \frac{a + \sqrt{a^2 - b}}{2}, \tag{5}$$

$$y = \frac{a - \sqrt{a^2 - b}}{2}. \tag{6}$$

Let, now, $a^2 - b$ be a perfect square, whose root is c; and substituting in (5) and (6), we have

$$x = \frac{a + c}{2}, \text{ and } y = \frac{a - c}{2}.$$

Substituting these values of x and y in (1) and (2), we obtain

$$\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + c}{2}} + \sqrt{\frac{a - c}{2}}; \tag{7}$$

$$\sqrt{a - \sqrt{b}} = \sqrt{\frac{a + c}{2}} - \sqrt{\frac{a - c}{2}}. \tag{8}$$

Hence, the square root of a binomial of the form of $a \pm \sqrt{b}$ may be found, when $a^2 - b$ is a perfect square, and (7) and (8) are formulas which apply to any particular example, by substituting the particular values for a, b, and c.

EXAMPLES.

1. Required the square root of $3 + 2\sqrt{2}$.

Here $a = 3$, $\sqrt{b} = 2\sqrt{2} = \sqrt{8}$, or $b = 8$, and $c = \sqrt{9 - 8} = 1$. Hence,

$$\sqrt{3 + 2\sqrt{2}} = \sqrt{\frac{3 + 1}{2}} + \sqrt{\frac{3 - 1}{2}} = \sqrt{2} + 1, \text{ Ans.}$$

2. Required the square root of $11 + \sqrt{72}$.

Ans. $3 + \sqrt{2}$.

3. Required the square root of $6 + \sqrt{20}$.

Ans. $1 + \sqrt{5}$.

4. Required the square root of $1 + 4\sqrt{-3}$.

Ans. $2 + \sqrt{-3}$.

5. Required the square root of $43 - 15\sqrt{8}$.

Ans. $5 - 3\sqrt{2}$.

6. Required the square root of $12 + 2\sqrt{35}$.

Ans. $\sqrt{5} + \sqrt{7}$.

7. Required the square root of $2m + 2\sqrt{m^2 - n^2}$.

Ans. $\sqrt{m+n} + \sqrt{m-n}$.

8. Required the square root of $-2\sqrt{-1}$.

Here a is equal to 0, and the required root is $1 - \sqrt{-1}$.

9. Required the square root of $\frac{a^2}{4} + \frac{b}{2}\sqrt{a^2 - b^2}$.

Ans. $\frac{b}{2} + \frac{1}{2}\sqrt{a^2 - b^2}$.

RADICAL EQUATIONS.

266. Radical Equations are those containing radical quantities.

267. The *solution* of a radical equation consists in rationalizing the terms containing the unknown quantity, and in determining its value.

The following examples containing radical quantities reduce to *simple* equations.

1. Given $\sqrt{x} + 1 = \sqrt{x-3} + 2$, to find the value of x.

OPERATION.

$$\sqrt{x} + 1 = \sqrt{x-3} + 2$$

Transposing and uniting, $\sqrt{x} = \sqrt{x-3} + 1$

Squaring, $x = x - 3 + 2\sqrt{x-3} + 1$

Or, $x = x - 2 + 2\sqrt{x-3}$

Transposing, $2 = 2\sqrt{x-3}$

Dividing by 2, $1 = \sqrt{x-3}$

Squaring, $1 = x - 3$

Whence, $x = 4$

2. Given $\frac{\sqrt{x}+38}{\sqrt{x}+6} = \frac{\sqrt{x}+28}{\sqrt{x}+4}$, to find the value of x.

OPERATION.

$$\frac{\sqrt{x}+38}{\sqrt{x}+6} = \frac{\sqrt{x}+28}{\sqrt{x}+4}$$

Clearing of fractions, $x+42\sqrt{x}+152 = x+34\sqrt{x}+168$

Transposing and uniting, $8\sqrt{x} = 16$

Or, $\sqrt{x} = 2$

Squaring, $x = 4$

3. Given $x\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1} = x+6$, to find the value of x.

OPERATION.

$$x\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1} = x+6$$

Multiplying both terms of the fraction by $\sqrt{1+x}+1$,

$$\frac{x(1+x+2\sqrt{1+x}+1)}{1+x-1} = x+6$$

Or, $2+x+2\sqrt{x+1} = x+6$

Transposing and uniting, $2\sqrt{x+1} = 4$

Or, $x+1 = 4$

Whence, $x = 3$

4. Given $\left\{x+(4ax+17a^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = x^{\frac{1}{2}}+a^{\frac{1}{2}}$, to find the value of x.

OPERATION.

$$\left\{x+(4ax+17a^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = x^{\frac{1}{2}}+a^{\frac{1}{2}}$$

Squaring, $x+(4ax+17a^2)^{\frac{1}{2}} = x+2a^{\frac{1}{2}}x^{\frac{1}{2}}+a$

Or, $(4ax+17a^2)^{\frac{1}{2}} = 2a^{\frac{1}{2}}x^{\frac{1}{2}}+a$

Squaring, $4ax+17a^2 = 4ax+4a^{\frac{3}{2}}x^{\frac{1}{2}}+a^2$

Transposing and uniting, $16\,a^2 = 4\,a^{\frac{3}{2}}x^{\frac{1}{2}}$

Dividing by $4\,a^{\frac{3}{2}}$, $4\,a^{\frac{1}{2}} = x^{\frac{1}{2}}$

Whence, by squaring, $x = 16\,a.$

5. Given $\dfrac{x-1}{\sqrt{x}+1} = 1 + \dfrac{\sqrt{x}-1}{2}$, to find the value of x.

OPERATION.

$$\frac{x-1}{\sqrt{x}+1} = 1 + \frac{\sqrt{x}-1}{2}$$

Simplifying the first member, $\sqrt{x} - 1 = 1 + \dfrac{\sqrt{x}-1}{2}$

Multiplying by 2, $2\sqrt{x} - 2 = 2 + \sqrt{x} - 1$

Transposing and uniting, $\sqrt{x} = 3$

Whence, by squaring, $x = 9.$

The foregoing illustrations furnish the following general suggestions: —

1. *Transpose the terms of the given equation so that a radical expression may stand alone in one member; then involve each member to a power of the same degree as the radical.*

2. *If there is still a radical expression remaining, the process of involution must be repeated.*

3. *Simplify as much as possible before performing the involution.*

Radicals may sometimes be removed by multiplying or dividing by a radical expression; hence they sometimes disappear on clearing of fractions. It is also occasionally convenient to rationalize the denominator of a fraction before removing denominators or involving.

EXAMPLES.

6. Given $\sqrt{x} + 6 = \sqrt{12 + x}$, to find x.

Ans. $x = 4.$

7. Given $\sqrt{x} + \sqrt{x-7} = \frac{21}{\sqrt{x-7}}$, to find x.

Ans. $x = 16$.

8. Given $\sqrt[3]{3x+4} + 3 = 6$, to find x.

Ans. $x = 7\frac{2}{3}$.

9. Given $\frac{x}{\sqrt{b^2+x}-b} = c$, to find x.

Ans. $x = c^2 - 2bc$.

10. Given $(x+2)^{\frac{1}{2n}} = (x^2+10x+1)^{\frac{1}{2n}}$, to find x.

Ans. $x = \frac{1}{2}$.

11. Given $\sqrt{x-32} = 16 - \sqrt{x}$, to find x.

Ans. $x = 81$.

12. Given $\sqrt{\frac{b}{a+x}} + \sqrt{\frac{c}{a-x}} = \sqrt[4]{\frac{4bc}{a^2-x^2}}$, to find x.

Ans. $x = a\left(\frac{b+c}{b-c}\right)$.

13. Given $a + x = \sqrt{a^2 + x\sqrt{b^2+x^2}}$, to find x.

Ans. $x = \frac{b^2}{4a} - a$.

14. Given $(x^{\frac{1}{2}} + 3a^2)^{\frac{1}{2}} - (x^{\frac{1}{2}} - 3a^2)^{\frac{1}{2}} = \frac{2a^{\frac{1}{2}}x^{\frac{1}{4}}}{b^{\frac{1}{2}}}$, to find x.

Ans. $x = \frac{9a^3b^2}{4(b-a)}$.

QUADRATIC EQUATIONS.

268. A QUADRATIC EQUATION is an equation of the *second degree* (Art. 146), or one in which the *square* is the highest power of the unknown quantity; as,

$$ax^2 = b, \text{ and } x^2 + 8x = 20.$$

Quadratic equations are divided into two classes, *Pure* and *Affected.*

PURE QUADRATIC EQUATIONS.

269. A PURE QUADRATIC EQUATION is one which contains only the square of the unknown quantity; as,

$$c\,x^2 = a\,b, \text{ and } x^2 = 400.$$

Equations of this kind are sometimes called *incomplete equations* of the second degree.

270. A pure quadratic equation can always, by transposition and reduction, be exhibited under the general form

$$x^2 = a,$$

in which a may represent any quantity, real or imaginary, positive or negative, integral or fractional.

Thus, the equation

$$\frac{20\,x^2}{3} - 5\,x^2 - 4 = \frac{5\,x^2}{4} + 2\frac{2}{3},$$

by clearing of fractions and transposing, reduces to

$$x^2 = 16.$$

Equations of this kind, therefore, have sometimes been denominated *binomial*, or those of *two terms*.

271. A binomial quadratic equation in the form

$$x^2 = a \qquad (1)$$

is readily solved, the value of x being evidently found by taking the square root of both members. Thus,

$$x = \pm\sqrt{a}, \qquad (2)$$

where the double sign is used because the square root of a quantity may be either positive or negative (Art. 213). The sign of x may also be negative, but still x will be either equal to $+\sqrt{a}$, or $-\sqrt{a}$. Either of these roots substituted in equation (1) will satisfy it. Hence

A pure quadratic equation has two roots, equal in numerical value, but with opposite signs.

1. Given $3x^2 + 7 = \frac{5x^2}{4} + 35$, to find x.

OPERATION.

$$3x^2 + 7 = \frac{5x^2}{4} + 35$$

Clearing of fractions, $12x^2 + 28 = 5x^2 + 140$

Transposing and uniting, $7x^2 = 112$

Or, $x^2 = 16$

Extracting the square root, $x = \pm 4$.

From the foregoing illustrations is deduced the following

RULE.

Reduce the given equation to the form $x^2 = a$, *and then extract the square root of both members.*

EXAMPLES.

2. Given $4x^2 - 7 = 29$, to find the values of x.

Ans. $x = \pm 3$.

3. Given $5x^2 + 5 = 3x^2 + 55$, to find the values of x.

Ans. $x = \pm 5$.

4. Given $7x^2 - 5 = 3x^2 + 11$, to find the values of x.

Ans. $x = \pm 2$.

5. Given $\frac{5}{4+x} + \frac{5}{4-x} = \frac{8}{3}$, to find the values of x.

Ans. $x = \pm 1$.

6. Given $x^2 - ab = d$, to find the values of x.

Ans. $x = \pm \sqrt{d + ab}$.

7. Given $ax^2 + n = m - c$, to find the values of x.

Ans. $x = \pm \sqrt{\frac{m - c - n}{a}}$.

8. Given $\frac{x^2}{2} - 3 + \frac{5x^2}{12} = \frac{7}{24} - x^2 + \frac{335}{24}$, to find the values of x. Ans. $x = \pm 3$.

9. Given $13 - \sqrt{3x^2 + 16} = 5$, to find the values of x. Ans. $x = \pm 4$.

10. Given $x + \sqrt{x^2 + a} = \frac{2a}{\sqrt{x^2 + a}}$, to find the values of x. Ans. $x = \pm \frac{1}{3}\sqrt{3a}$.

11. Given $\frac{1}{1 - \sqrt{1 - x^2}} - \frac{1}{1 + \sqrt{1 - x^2}} = \frac{\sqrt{3}}{x^2}$, to find the values of x. Ans. $x = \pm \frac{1}{2}$.

12. Given $\frac{\sqrt{a + x}}{\sqrt{x}} + \frac{\sqrt{a - x}}{\sqrt{x}} = \sqrt{\frac{x}{6}}$, to find the values of x. Ans. $x = \pm \sqrt{24a - 144}$.

272. Simultaneous Equations (Art. 163) sometimes produce pure equations after elimination. The methods of elimination are the same as in simple equations.

13. Given $x^2 - 9y^2 = 63$, and $\frac{x}{2} = -2y$, to find the values of x and y.

OPERATION.

$$x^2 - 9y^2 = 63 \quad (1)$$

$$\frac{x}{2} = -2y \quad (2)$$

Clearing (2) of fractions, $x = -4y$ (3)

Squaring (3), $x^2 = 16y^2$ (4)

Substituting the value of x^2 in (1), $16y^2 - 9y^2 = 63$ (5)

Or, $7y^2 = 63$

Whence, $y^2 = 9$

Extracting the square root, $y = \pm 3$

Substituting the value of y in (3), $x = \mp 12$.

14. Given $3x + 3y = 5x$, and $(x + y)x = 960$, to find the values of x and y.

Ans. $x = \pm 24$; $y = \pm 16$.

15. Given $3x^2 + 4y^2 = 76$, and $3y^2 - 11x^2 = 4$, to find the values of x and y.

Ans. $x = \pm 2$; $y = \pm 4$.

16. Given $x^2 - xy = 54$, and $xy - y^2 = 18$, to find the values of x and y. Ans. $x = \pm 9$; $y = \pm 3$.

17. Given $x^2 + xy = 60$, and $xy + y^2 = 84$, to find the values of x and y. Ans. $x = \pm 5$; $y = \pm 7$.

18. Given $x^2 - y^2 = 2268$, and $\frac{63x^2}{4y} = 28y$, to find the values of x and y. Ans. $x = \pm 72$; $y = \pm 54$.

PROBLEMS

LEADING TO PURE EQUATIONS.

1. Bought a lot of flour for $175, and the number of dollars per barrel was to the number of barrels as 4 to 7. How many barrels were purchased, and what was the price of each?

SOLUTION.

Let x = the no. of dollars per barrel,

then $\frac{7x}{4}$ = the number of barrels.

Therefore, $\frac{7x^2}{4} = 175$

Clearing of the fraction, $7x^2 = 700$

Or, $x^2 = 100$

Extracting the square root, $x = 10$, no. of dollars per barrel,

whence, $\frac{7x}{4} = 17\frac{1}{2}$, the number of barrels.

2. I have three square house-lots, of equal size. If I were to add 193 square rods to their contents, they would be equivalent to a square lot whose sides would each measure 25 rods. Required the length of each side of the three lots. Ans. 12 rods.

3. A and B lay out money on speculation; the amount of A's stock and gain is $27, and he gains as much per cent. on his stock as B lays out. B's gain is $32; and it appears that A gains twice as much per cent. as B. Required the capital of each.

Ans. A's capital, $15; B's, $80.

4. There are two square fields, the larger of which contains 25,600 square rods more than the other, and the ratio of their sides is as 5 to 3. Required the contents of each.

Ans. The larger, 40,000 square rods; the smaller, 14,400 square rods.

5. Bought sugar at such a rate, that the price of a pound was to the number of pounds as 4 to 5. If the cost of the whole had been 45 cents more, the number of pounds would have been to the price of a pound as 4 to 5. How many pounds were bought, and what was the price per pound?

SOLUTION.

Let x = the no. of pounds,
and y = the price per pound.
Then xy = the whole cost.

Therefore, $$y = \frac{4x}{5} \quad (1)$$

And $$x = \frac{4xy + 180}{5x} \quad (2)$$

Or, $$5x^2 = 4xy + 180 \quad (3)$$

Substituting value of y in (3), $$5x^2 = \frac{16x^2}{5} + 180$$

Or, $$25x^2 = 16x^2 + 900$$

Transposing and uniting, $$9x^2 = 900$$

Or, $$x^2 = 100$$

Extracting the square root, $x = 10$, the no. of pounds.

Substituting the value of x in (1), $y = 8$, the price per pound.

6. A and B engage in a speculation. A disposes of his share for $11, and gains as many per cent. as B

invested dollars. B's gain was $36, and the gain upon A's investment was 4 times as many per cent. as upon B's. How much did each invest?

Ans. A invested $5; B, $120.

7. What two numbers are those whose difference multiplied by the less is 42, or by their sum is 133?

Ans. ± 13 and ± 6.

8. Two workmen, A and B, engage for a certain number of weeks, at different rates. At the end of the time, A, who had been absent 4 weeks, received $75; but B, who had been absent 7 weeks, received only $48. Now, if B had been absent only 4 weeks, and A 7 weeks, they would have received exactly alike. How many weeks were they engaged for, how many did each work, and what had each per week?

Ans. Engaged for 19 weeks; A worked 15 weeks, B 12 weeks; A received $5, B $4, per week.

Here, the conditions lead to the equation $25(x-7)^2 = 16(x-4)^2$, which, by taking the square root of each member, gives $5(x-7) = 4(x-4)$.

AFFECTED QUADRATIC EQUATIONS.

273. An Affected Quadratic Equation is one which contains both the second and first powers of the unknown quantity; as,

$$x^2 + ax = b, \text{ and } ax^2 + bx - c = bx^2 - ax + d.$$

Equations of this kind, containing every power of the unknown quantity from the first to the highest given, are sometimes called *complete equations.*

274. Any affected quadratic equation can always, by reduction, be exhibited under the general form

$$x^2 + px = q,$$

where p and q are understood to represent any quantities, positive or negative, integral or fractional.

For we can reduce it to this form, by bringing all the terms containing the unknown quantity to the first member of the equation, and the known quantities to the other, and then dividing by the coefficient of x^2.

Equations of this kind, therefore, have sometimes been denominated *trinomial*, or those of *three terms*.

275. To solve an affected quadratic equation, it is obviously required to take the square root of its members. But the member containing the unknown quantities alone cannot be a perfect square (Art. 224), nor can it often be so rendered by transposition. It is therefore necessary to have recourse to methods of completing its square.

FIRST METHOD OF COMPLETING THE SQUARE.

276. Suppose we have given any quadratic equation,

$$x^2 + px = q,$$

in which it is required to find the values of x.

Adding $\left(\frac{p}{2}\right)^2$ to both members (Art. 90), we have

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4},$$

where the first member is a complete square. Extracting the square root, we obtain

$$x + \frac{p}{2} = \pm\sqrt{q + \frac{p^2}{4}},$$

Transposing,

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

Here, since px must be twice the product of the two terms of the root of the completed square (Art. 90),

and x being one of those terms, $\frac{p}{2}$ must be the other. We therefore made $x^2 + px$ a perfect square by adding to it $\left(\frac{p}{2}\right)^2$, or $\frac{p^2}{4}$, which had also to be added to the second member to preserve the equality.

The double sign is used in the result, since the square of both $+\sqrt{q+\frac{p^2}{4}}$ and $-\sqrt{q+\frac{p^2}{4}}$ is $+\left(q+\frac{p^2}{4}\right)$; consequently,

Every quadratic equation may have two roots.

1. Given $2x^2 + 12x + 36 = 356$, to find the values of x.

OPERATION.

$$2x^2 + 12x + 36 = 356$$

By transposition, $2x^2 + 12x = 320$

Dividing by 2, $x^2 + 6x = 160$

Completing the square, $x^2 + 6x + 9 = 169$

Extracting the square root, $x + 3 = \pm 13$

Transposing, $x = -3 \pm 13$

Whence, $x = 10$, or -16.

Here, taking 13, the positive root of 169, we find $x = 10$; but taking -13, the negative root, we find $x = -16$.

2. Given $x^2 - 12x + 30 = 3$, to find the values of x.

OPERATION.

$$x^2 - 12x + 30 = 3$$

By transposition, $x^2 - 12x = -27$

Completing the square, $x^2 - 12x + 36 = 9$

Extracting the square root, $x - 6 = \pm 3$

Transposing, $x = 6 \pm 3$

Whence, $x = 9$, or 3;

where both values of x are positive.

3. Given $-3x^2-7x=\frac{10}{3}$, to find the values of x.

OPERATION.

$$-3x^2-7x=\tfrac{10}{3}$$

Dividing by -3, $\quad x^2+\tfrac{7}{3}x=-\tfrac{10}{9}$

Completing the square, $\quad x^2+\tfrac{7}{3}x+\tfrac{49}{36}=\tfrac{9}{36}$

Extracting the square root, $\quad x+\tfrac{7}{6}=\pm\tfrac{3}{6}$

Transposing, $\quad x=-\tfrac{7}{6}\pm\tfrac{3}{6}$

Whence, $\quad x=-\tfrac{2}{3}$, or $-\tfrac{5}{3}$;

where both values of x are negative.

The results obtained in each of these operations may be readily verified. Hence the following

GENERAL RULE.

Reduce the given equation to the form $x^2+px=q$. *Complete the square by adding the square of half the coefficient of* x *to both members. Extract the square root of both members, and solve the simple equation thus produced.*

Sometimes the first member of the equation reduces to a perfect square, and the second to 0. Then the root may be found directly; but x will really have but one value. Thus,

$$x^2-6x=-9$$

by transposition becomes

$$x^2-6x+9=0,$$

where the first member is the square of $x-3$, whence

$$x=3\pm 0=3.$$

It is, however, convenient in this case to consider the quadratic equation as having *two equal roots.*

EXAMPLES.

4. Given $x^2 - 6x + 12 = 4$, to find the values of x.

Ans. $x = 4$, or 2.

5. Given $x^2 - 8x + 50 = 98$, to find the values of x.

Ans. $x = 12$, or -4.

6. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$, to find the values of x.

Ans. $x = \frac{2}{3}$, or $\frac{1}{3}$.

7. Given $-3x^2 + 36x - 105 = 0$, to find the values of x.

Ans. $x = 7$, or 5.

8. Given $4x - \frac{36 - x}{x} = 46$, to find the values of x.

Ans. $x = 12$, or $-\frac{3}{4}$.

9. Given $6x - 30 = 3x^2$, to find the values of x.

Ans. $x = 1 \pm 3\sqrt{-1}$.

10. Given $\frac{x^2}{2} - \frac{x}{3} + 20\frac{1}{2} = 42\frac{2}{3}$, to find the values of x.

Ans. $x = 7$, or $-6\frac{1}{3}$.

11. Given $x^2 + ax = b$, to find the values of x.

Ans. $x = -\frac{a}{2} + \sqrt{b + \frac{a^2}{4}}$, or $-\frac{a}{2} - \sqrt{b + \frac{a^2}{4}}$.

277. Since the equation $x^2 + px = q$ is the general expression of any affected quadratic equation reduced to that form (Art. 276), we may use its roots,

$$-\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}, \text{ and } -\frac{p}{2} - \sqrt{q + \frac{p^2}{4}},$$

as the general formulas for the roots of any affected quadratic equation, and write out its roots, by substituting the particular values of p and q in these formulas. Hence,

Write for the roots of an equation in the form of $x^2 + px = q$, *half the coefficient of* x *with a contrary sign, plus or minus the square root of the sum of the second member and the square of said half coefficient.*

12. Given $x^2 + 4x = 140$, to find the values of x.

Here $\frac{p}{2} = 2$, and $q = 140$, whence, by substitution,

$$x = -2 \pm \sqrt{140 + 4} = 10, \text{ or } -14.$$

13. Given $2x^2 + 8x - 20 = 70$, to find the values of x.

Ans. $x = 5$, or -9.

14. Given $2x^2 - 2x - \frac{3}{2} = 0$, to find the values of x.

Ans. $x = \frac{3}{2}$, or $-\frac{1}{2}$.

15. Given $x^2 + 6x + 2 = -6$, to find the values of x.

Ans. $x = -2$, or -4.

16. Given $5x - \frac{3x-3}{x-3} = 2x + \frac{3x-6}{2}$, to find the values of x. Ans. $x = 4$, or -1.

SECOND METHOD OF COMPLETING THE SQUARE.

278. Any affected quadratic equation may be solved by the first method, since its rule is general; but it is sometimes more convenient to employ another mode of completing the square, known as the "Hindoo method."

An affected quadratic reduced to three terms, cleared of all fractions, and divided by the greatest common divisor of its terms, may thus be reduced to the form

$$ax^2 + bx = c,$$

where a, b, and c represent any whole numbers whatever which have no common divisor greater than unity.

To make the first term of this equation a square, and the second term divisible by 2, we multiply both members by $4a$, and have

$$4a^2x^2 + 4abx = 4ac.$$

Now, $2ax$, the square root of $4a^2x^2$, must be the first term of the root, and $\frac{4abx}{2}$, or $2abx$, must be the prod-

uct of its two terms; hence, $\frac{2abx}{2ax}$, or b, must be the second term of the root; therefore b^2 must be added to complete the square, giving

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2,$$

where the first member is a complete square, equal to $(2ax + b)^2$. Whence, taking the square root, we have

$$2ax + b = \pm\sqrt{4ac + b^2},$$

or, by transposing, $$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

It will be observed that the quantity added in completing the square is the square of the coefficient of x in the original equation; and that the introduction of fractions has been avoided in completing the square.

As a, b, and c in the equation may have any value whatever, the process is general. Hence the

RULE.

Reduce the equation to the form $ax^2 + bx = c$, *in which* a, b, *and* c *are prime to each other.*

Multiply both members of the equation by FOUR *times the coefficient of* x^2, *and add to each the square of the coefficient of* x.

Extract the square root of both members, and solve the simple equation thus produced.

Any quadratic equation may also be solved by using the *coefficient* of x^2 as a multiplier, instead of four times that coefficient, and adding the square of *one half* the coefficient of x, instead of the square of that coefficient. If the coefficient of x is an even number, this method will avoid fractions in completing the square.

The formula $x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}$ may be used for the solution of an equation in the form $ax^2 + bx = c$, in the same manner as the formula $x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}$ is used in Article 277.

EXAMPLES.

1. Given $x^2 - 5x + 4 = 0$, to find the values of x.

OPERATION.

$$x^2 - 5x + 4 = 0$$

By transposition, $x^2 - 5x = -4$

Multiplying by 4 times the coefficient of x^2, we have $4x^2 - 20x = -16$

Adding to each member the square of the coefficient of x, $4x^2 - 20x + 25 = 9$

Extracting the square root, $2x - 5 = \pm 3$

Or, $2x = 5 \pm 3 = 8$, or 2

Whence, $x = 4$, or 1.

2. Given $3x^2 = 42 - 5x$, to find the values of x.

Ans. $x = 3$, or $-4\frac{2}{3}$.

3. Given $\sqrt{(4 + x)(5 - x)} = 2x - 10$, to find the values of x.

Ans. $x = 5$, or $3\frac{1}{3}$.

4. Given $\frac{6}{x + 1} + \frac{2}{x} = 3$, to find the values of x.

Ans. $x = 2$, or $-\frac{1}{3}$.

5. Given $2x^2 - 7x + 3 = 0$, to find the values of x.

Ans. $x = 3$, or $\frac{1}{2}$.

6. Given $acx^2 - bcx + adx = bd$, to find the values of x.

Ans. $x = \frac{b}{a}$, or $-\frac{d}{c}$.

7. Given $110x^2 - 21x + 1 = 0$, to find the values of x.

Ans. $x = \frac{1}{10}$, or $\frac{1}{11}$.

8. Given $125x^2 - 7x = 17\frac{1}{5}$, to find the values of x.

Ans. $x = \frac{2}{5}$, or $-\frac{43}{125}$.

9. Given $ax^2 - 2bx = c$, to find the values of x.

Ans. $x = \frac{b}{a} \pm \frac{1}{a}\sqrt{ac + b^2}$.

THIRD METHOD OF COMPLETING THE SQUARE.

279. An affected quadratic may be exhibited under the form

$$a^2x^2 + bx = c, \qquad (1)$$

when the coefficient of the highest power of the unknown quantity is a perfect square.

Adding some quantity, as m^2, to the first member to complete its square, we have

$$a^2x^2 + bx + m^2 = c + m^2. \qquad (2)$$

Now, from the nature of the square of a binomial (Art. 90), twice the product of the roots of the extreme terms must equal the second term. Therefore,

$$2\,(ax \times m) = bx,$$

whence $$m = \frac{b}{2a}, \text{ and } m^2 = \left(\frac{b}{2a}\right)^2.$$

Substituting the value of m in (2), we obtain

$$a^2x^2 + bx + \left(\frac{b}{2a}\right)^2 = c + \left(\frac{b}{2a}\right)^2,$$

where the square is complete. Hence the

RULE.

Reduce the given equation to the form $a^2x^2 + bx = c$, *either by multiplication or division. Divide then the second*

term by twice the square root of the first, and add the square of the quotient to both members of the equation.

Extract the square root of both members, and solve the simple equation thus produced.

EXAMPLES.

1. Given $4x^2 + 16x = 33$, to find the values of x.

Ans. $x = \frac{3}{2}$, or $-\frac{11}{2}$.

$\sqrt{4x^2} = 2x$; $\frac{16x}{4x} = 4$; and $4^2 = 16$. Hence, the addition of 16 will complete the square.

2. Given $3x^2 + 2x + 6 = 11$, to find the values of x.

Ans. $x = 1$, or $-\frac{5}{3}$.

3. Given $16x^2 + 15x = 34$, to find the values of x.

Ans. $x = \frac{17}{16}$, or -2.

4. Given $9x^{-2} - 12x^{-1} = -3$, to find the values of x.

Ans. $x = 1$, or 3.

5. Given $\frac{(2x-3)(2x-3)}{x} = 8$, to find the values of x.

Ans. $x = \frac{9}{2}$, or $\frac{1}{2}$.

6. Given $\frac{9x^2}{16} + \frac{30x}{8} = -4$, to find the values of x.

Ans. $x = -1\frac{1}{3}$, or $-5\frac{1}{3}$.

7. Given $\frac{4x^2}{49} + \frac{16x}{42} - \frac{20}{3} = 0$, to find the values of x.

Ans. $x = 7$, or $-11\frac{2}{3}$.

280. The following equations may be solved by either of the preceding methods, preference being given to the one best adapted to the example considered. Special methods and devices may also be employed whenever any advantage can thereby be gained.

Examples.

1. Given $x^2 - x + 3 = 45$, to find the values of x.
   Ans. $x = 7$, or -6.

2. Given $(x - 12)(x + 2) = 0$, to find the values of x.
   Ans. $x = 12$, or -2.

3. Given $2x^2 - x = 21$, to find the values of x.
   Ans. $x = \frac{7}{2}$, or -3.

4. Given $4x - \frac{14 - x}{x + 1} = 14$, to find the values of x.
   Ans. $x = 4$, or $-\frac{7}{4}$.

5. Given $\frac{(x-1)^2}{2} + \frac{x-1}{4} = 9$, to find the values of x.

OPERATION.

$$\frac{(x-1)^2}{2} + \frac{x-1}{4} = 9$$

Multiplying by 2, $(x-1)^2 + \frac{1}{2}(x-1) = 18$

Considering $(x-1)^2$ a simple quantity, and completing the square,

$$(x-1)^2 + \tfrac{1}{2}(x-1) + \tfrac{1}{16} = 18 + \tfrac{1}{16} = \tfrac{289}{16}$$

Extracting the square root, $(x-1) + \frac{1}{4} = \pm \frac{17}{4}$

Or, $x - \frac{3}{4} = \pm \frac{17}{4}$

Whence, $x = \frac{3}{4} \pm \frac{17}{4}$

Or, $x = 5$, or $-\frac{7}{2}$.

6. Given $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$, to find the values of x.
   Ans. $x = 3$, or $2\frac{1}{11}$.

7. Given $x + \sqrt{5x + 10} = 8$, to find the values of x.
   Ans. $x = 18$, or 3.

8. Given $x + \sqrt{10x + 6} = 9$, to find the values of x.
   Ans. $x = 25$, or 3.

9. Given $48x^{-2} + 32x^{-1} = 11$, to find the values of x.
   Ans. $x = 4$, or $-1\frac{1}{11}$.

10. Given $81x^2 - 36x + 4 = 49$, to find the values of x. Ans. $x = 1$, or $-\frac{5}{9}$.

11. Given $\sqrt{3x-5} = \frac{\sqrt{7x^2+36x}}{x}$, to find the values of x. Ans. $x = 6$, or -2.

12. Given $\frac{x^2}{5} + 20x = 3x^2 - 80$, to find the values of x. Ans. $x = 10$, or $-2\frac{6}{7}$.

13. Given $\frac{x-1}{x+1} + \frac{x+3}{x-3} = 2\left(\frac{x+2}{x-2}\right)$, to find the values of x. Ans. $x = 5$, or 0.

14. Given $\frac{x}{2} + \frac{2}{x} = \frac{1}{3} + \frac{3}{x}$, to find the values of x.

Ans. $x = \frac{1 \pm \sqrt{19}}{3}$.

15. Given $\frac{21}{5-x} - \frac{x}{7} = 3\frac{2}{7}$, to find the values of x.

Ans. $x = -2$, or -16.

16. Given $x^2 - 2ax + a^2 - b^2 = 0$, to find the values of x. Ans. $x = a \pm b$.

17. Given $6x - 33 = 3x^2$, to find the values of x.

Ans. $x = 1 \pm \sqrt{-10}$.

18. Given $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$, to find the values of x. Ans. $x = -a$, or $-b$.

19. Given $x^2 - 2ax + b^2 = 0$, to find the values of x.

Ans. $x = a \pm \sqrt{a^2 - b^2}$.

20. Given $\frac{2x(a-x)}{3a-2x} = \frac{a}{4}$, to find the values of x.

Ans. $x = \frac{3a}{4}$, or $\frac{a}{2}$.

21. Given $6x + \frac{35-3x}{x} - 44 = 0$, to find the values of x. Ans. $x = 7$, or $\frac{5}{6}$.

22. Given $3x = 10 + \frac{x^2}{4}$, to find the values of x.

Ans. $x = 6 \pm 2\sqrt{-1}$.

23. Given $\frac{3}{4}(x^2 - 3) = \frac{x-3}{8}$, to find the values of x.

Ans. $x = 1\frac{2}{3}$, or $-1\frac{1}{2}$.

24. Given $x^2 - 4ax + a^2 + b^2 = 0$, to find the values of x. Ans. $x = 2a \pm \sqrt{3a^2 - b^2}$.

25. Given $3x - \frac{x^2}{4} + 3 = 14$, to find the values of x.

Ans. $x = 6 \pm \sqrt{-8}$.

26. Given $x^2 + bx + cx = a^2 - ab + ac - bc$, to find the values of x. Ans. $x = a - b$, or $-a - c$.

27. Given $abx^2 + \frac{3a^2x}{c} = \frac{6a^2 + ab - 2b^2}{c^2} - \frac{b^2x}{c}$, to find the values of x. Ans. $x = \frac{2a-b}{ac}$, or $-\frac{3a+2b}{bc}$.

28. Given $x^2 - 8x = 14$, to find the values of x.

Ans. $x = 9.4772+$, or $-1.4772+$.

29. Given $(7 - 4\sqrt{3})x^2 + (2 - \sqrt{3})x = 2$, to find the values of x.

Ans. $x = 2 + \sqrt{3}$, or $-2(2 + \sqrt{3})$.

30. Given $\frac{x+a}{x-2a} + \frac{x-2a}{x+a} = 1$, to find the values of x.

Ans. $x = \frac{a}{2}\left(1 \pm \sqrt{-27}\right)$.

PROBLEMS

LEADING TO AFFECTED QUADRATIC EQUATIONS.

1. There is a certain number whose square increased by 30 is equal to 11 times the number itself. Required the number.

SOLUTION.

Let x = the number.

Then, $x^2 + 30 = 11x$

Or, $x^2 - 11x = -30$

Completing the square, $x^2 - 11x + \frac{121}{4} = \frac{1}{4}$

Whence, $x - \frac{11}{2} = \pm \frac{1}{2}$

And $x = 6$, or 5.

The number is either 6 or 5, for each of these values satisfies the conditions of the problem.

2. Find two numbers such that their sum shall be 15, and the sum of their squares 117. Ans. 6 and 9.

3. A person cut and piled two ranges of wood, whose united contents were 26 cords, for 356 dimes; and the labor on each of them cost as many dimes per cord as there were cords in its range. Required the number of cords in each range. Ans. 10 cords and 16 cords.

4. Bought a watch, which I sold for $56, and thereby gained as much per cent. as the watch cost me. Required the amount paid for it.

SOLUTION.

Let x = the amount paid, in dols.

Then x = the gain per cent.,

and $\frac{x}{100} \times x = \frac{x^2}{100}$ = the whole gain, in dollars.

Therefore, $\frac{x^2}{100} = 56 - x$

Or, $x^2 + 100x = 5600$

Completing the square,

$x^2 + 100x + 2500 = 8100$

Whence, $x + 50 = \pm 90$

And $x = 40$, or -140.

Only the positive value of x is here admissible, as the negative result does not answer to the conditions of the problem. The cost, therefore, was $40.

5. A grazier bought a certain number of oxen for $240, and having lost 3, he sold the remainder at $8 a head more than they cost him, thus gaining $59 by his bargain. What number did he buy? Ans. 16.

6. The plate of a rectangular looking-glass is 18 inches by 12, and is to be framed with a frame all parts of which are of equal width, and whose area is to be equal to that of the glass. Required the width of the frame.

Ans. 3 inches.

7. A merchant sold a quantity of flour for $39, and gained as much per cent. as the flour cost him. What was the cost of the flour? Ans. $30.

8. There are two numbers, whose difference is 9, and whose sum multiplied by the greater is 266. What are those numbers? Ans. 14 and 5.

9. A and B gained by trade $1800. A's money was in the firm 12 months, and he received, for his principal and gain, $2600. B's money, which was $3000, was in the firm 16 months. What money did A put into the firm? Ans. $2000.

10. A merchant bought a quantity of flour for $72, and found that if he had bought 6 barrels more for the same money, he would have paid $1 less for each barrel. How many barrels did he buy, and what was the price of each?

Ans. 18 barrels, at $4 per barrel.

11. A square court-yard has a gravel-walk around it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 164 yards. Required the area of the court. Ans. 256 square yards.

12. My gross income is $1000. After deducting a percentage for income tax, and then a percentage less by one than that of the income tax from the remainder, the income

is reduced to $912. Required the rate per cent. at which the income tax is charged. Ans. 5 per cent.

13. I have a rectangular field of corn, which consists of 6250 hills, but the number of hills in the length exceeds the number in the breadth by 75. Of how many hills does the length and breadth consist?

Ans. Length, 125 hills; breadth, 50 hills.

14. A certain company agreed to build a vessel for $6300; but, two of their number having died, those that survived had each to advance $200 more than they otherwise would have done. Of how many persons did the company at first consist? Ans. 9 persons.

15. A detachment from an army was marching in regular column, with 6 men more in depth than in front; but when the enemy came in sight, the front was increased by 870 men, and the whole was thus drawn up in 4 lines. Required the number of men. Ans. 3712 men.

16. A has two square gardens, and the side of the one exceeds that of the other by 4 rods, while the contents of both are 208 square rods. How many square rods does the larger garden contain more than the smaller?

Ans. 80 square rods.

17. A certain farm is a rectangle, whose length is twice its breadth; but should it be enlarged 20 rods in length and 24 rods in breadth, its contents would be doubled. Of how many acres does the farm consist?

Ans. 20 acres.

18. A square court-yard has a rectangular gravel-walk around it. The side of the court wants one yard of being six times the breadth of the gravel-walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 340. What is the area of the court and width of the walk?

Ans. Area of the court, 529 square yards; width of the walk, 4 yards.

19. A merchant bought 54 bushels of wheat, and a certain quantity of barley. For the former he gave half as many dimes per bushel as there were bushels of barley, and for the latter 4 dimes per bushel less. He sold the mixture at \$1 per bushel, and lost \$ 57.60 by his bargain. Required the quantity of the barley, and its price.

Ans. 36 bushels, at \$1.40 per bushel.

20. A lady wishes to purchase a carpet for each of her square parlors; the side of one of them is 1 yard longer than the other, and it will require 85 square yards for both rooms. What will it cost the lady to carpet each of the rooms with carpeting 40 inches wide, at \$1.75 per yard?

Ans. Larger room. \$77.17½; smaller, \$56.70.

EQUATIONS IN THE QUADRATIC FORM.

281. An equation has the quadratic form when it is expressed in three terms, two of which contain the unknown quantity; and of these two, *one has an exponent twice as great as the other.* The quadratic form, then, is

$$x^{2n} + p\,x^n = q, \quad \text{or} \quad a\,x^{2n} + b\,x^n = c,$$

where n may have any value whatever, and x may also represent either the unknown quantity, or some expression containing it.

282. The rules already given for the solution of quadratics, will apply to equations having the same form.

For, in the equation, $x^{2n} + p\,x^n = q$, assume $y = x^n$, then

$$y^2 + p\,y = q.$$

Whence, by rule, Art. 277, we have

$$x^n = y = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

Extracting the nth root of both members,

$$x = \sqrt[n]{-\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}}.$$

EXAMPLES.

1. Given $x^4 - 5x^2 = -4$, to find the values of x.

FIRST OPERATION.

$$x^4 - 5x^2 = -4$$

Assuming $x^2 = y$, we have $y^2 - 5y = -4$

Completing the square, $y^2 - 5y + \frac{25}{4} = \frac{9}{4}$

Extracting the square root, $y - \frac{5}{2} = \pm \frac{3}{2}$

Whence, $y = 4$, or 1

But $x^2 = y$, therefore $x^2 = 4$, or 1

And $x = \pm 2$, or ± 1.

SECOND OPERATION.

$$x^4 - 5x^2 = -4$$

Completing the square, $x^4 - 5x^2 + \frac{25}{4} = \frac{9}{4}$

Extracting the square root, $x^2 - \frac{5}{2} = \pm \frac{3}{2}$

Or, $x^2 = 4$, or 1

Whence, $x = \pm 2$, or ± 1.

In the second operation a new letter, y, has not been introduced; but the principle of both operations is the same. Each gives x *four* values: $+2$, -2, $+1$, and -1.

2. Given $x^6 - 6x^3 = 16$, to find the values of x.

OPERATION.

$$x^6 - 6x^3 = 16$$

Completing the square, $x^6 - 6x^3 + 9 = 25$

Extracting the square root, $x^3 - 3 = \pm 5$

Or, $x^3 = 8$, or -2

Whence, $x = 2$, or $\sqrt[3]{-2}$.

Here, on completing the square and extracting the square root, we have a pure cubic equation, from which,

taking the root indicated, after the manner of solving pure equations (Art. 271), we obtain *two* values of x. It will hereafter be shown that equations of the sixth degree have four other roots; but these cannot be found by the present process.

3. Given $x^4 + 4x^2 = 117$, to find the values of x.

Ans. $x = \pm 3$, or $\pm \sqrt{-13}$.

4. Given $x^{-4} - 9x^{-2} + 20 = 0$, to find the values of x.

Ans. $x = \pm \frac{1}{5}\sqrt{5}$, or $\pm \frac{1}{2}$.

5. Given $x^{10} + 31x^5 - 10 = 22$, to find the values of x.

Ans. $x = 1$, or -2.

6. Given $81x^2 + x^{-2} = 82$, to find the values of x.

Ans. $x = \pm 1$, or $\pm \frac{1}{3}$.

Here it is convenient to complete the square by supplying the middle term, which, from the nature of a binomial square, must be twice the product of the square roots of $81x^2$ and x^{-2}, or 18.

7. Given $x^2 + \frac{1225}{x^2} - 14 = 60$, to find the values of x.

Ans. $x = \pm 7$, or ± 5.

8. Given $x^6 + 20x^3 - 10 = 59$, to find the values of x.

Ans. $x = \sqrt[3]{3}$, or $\sqrt[3]{-23}$.

9. Given $3x^{2n} - 2x^n = 25$, to find the values of x.

Ans. $x = \sqrt[n]{\frac{1 \pm 2\sqrt{19}}{3}}$.

283. Radical Equations sometimes take the quadratic form, and reduce to pure equations.

10. Given $x + 4\sqrt{x} = 21$, to find the values of x.

OPERATION.

$$x + 4\sqrt{x} = 21$$

Completing the square, $x + 4\sqrt{x} + 4 = 25$

Extracting the square root, $\sqrt{x} + 2 = \pm 5$

Or, $\sqrt{x} = 3$, or -7

Whence, $x = 9$, or 49

Here the exponent of the first term is ^{1}, or $^{\frac{2}{2}}$, and that of the second is $^{\frac{1}{2}}$, for $4\sqrt{x} = 4x^{\frac{1}{2}}$; hence the equation has the quadratic form. In verifying the values obtained for x, we find that 9 is limited to the positive square root, while 49 is limited to the negative square root, since those roots only will satisfy the equation. The real roots of the equation, then, are $(+3)^2$ and $(-7)^2$.

11. Given $3x^2 + x^{\frac{7}{6}} = 3104\,x^{\frac{1}{3}}$, to find the values of x.

OPERATION.

$$3x^2 + x^{\frac{7}{6}} = 3104\,x^{\frac{1}{3}}$$

Dividing by $x^{\frac{1}{3}}$, $3x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$

Multiplying by 12, $36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} = 37248$

Adding 1, $36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} + 1 = 37249$

Extracting the square root, $6x^{\frac{5}{6}} + 1 = \pm 193$

Or, $6x^{\frac{5}{6}} = 192$, or -194

Dividing by 6, $x^{\frac{5}{6}} = 32$, or $-\frac{97}{3}$

Extracting the fifth root, $x^{\frac{1}{6}} = 2$, or $(-\frac{97}{3})^{\frac{1}{5}}$

Whence, $x = 64$, or $(-\frac{97}{3})^{\frac{6}{5}}$.

Here, on dividing both members of the equation by $x^{\frac{1}{3}}$, we obtain an equation in the quadratic form, since $^{\frac{5}{3}}$ is twice as great as $^{\frac{5}{6}}$.

As we extract the corresponding root to remove the numerator of the exponent of x, and raise to a corresponding power to remove its denominator, the effect is the same as if we at once transfer the exponent of x to the second member by inverting it. That is,

The EXPONENT *of either member of an equation may be transferred to the other member by inverting it.*

It may also be here noted, that

A FACTOR *of either member of an equation may be transposed to the other member by changing the sign of its exponent.*

The factor and exponent, however, must belong to the *whole member*, not to a single term only, nor, in the case of the exponent, to a single factor only.

12. Given $3x^{\frac{4}{3}} - \frac{5x^{\frac{8}{3}}}{2} = -592$, to find the values of x.

Ans. $x = 8$, or $\sqrt[4]{(-\frac{14}{5})^8}$.

13. Given $x^3 - x^{\frac{3}{2}} = 56$, to find the values of x.

Ans. $x = 4$, or $\sqrt[3]{49}$.

14. Given $x^n - 2ax^{\frac{n}{2}} = b$, to find the values of x.

Ans. $x = (a \pm \sqrt{a^2 + b})^{\frac{2}{n}}$.

15. Given $\sqrt{x^3} - 2\sqrt{x} - x = 0\sqrt{x}$, to find the values of x. Ans. $x = 4$, or 1.

16. Given $\frac{\sqrt{4x} + 2}{4 + \sqrt{x}} = \frac{4 - \sqrt{x}}{\sqrt{x}}$, to find the values of x.

Ans. $x = 4$, or $\frac{64}{9}$.

17. Given $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$, to find the values of x.

Ans. $x = 243$, or $-28^{\frac{5}{3}}$.

18. Given $\frac{\frac{3\sqrt{x}}{5} - 2}{x - 5} = \frac{1}{20}$, to find the values of x.

Ans. $x = 49$, or 25.

284. Polynomials may sometimes take the place of the unknown quantity as the basis of the quadratic form.

19. Given $2x^2 + \sqrt{2x^2+1} = 11$, to find the values of x.

OPERATION.

$$2x^2 + \sqrt{2x^2+1} = 11$$

Adding 1 to both members, we have the quadratic form,

$$2x^2 + 1 + \sqrt{2x^2+1} = 12$$

Or, $$(2x^2+1) + (2x^2+1)^{\frac{1}{2}} = 12$$

Completing the square,

$$(2x^2+1) + (2x^2+1)^{\frac{1}{2}} + \tfrac{1}{4} = \tfrac{49}{4}$$

Extracting the square root,

$$(2x^2+1)^{\frac{1}{2}} + \tfrac{1}{2} = \pm \tfrac{7}{2}$$

Or, $$(2x^2+1)^{\frac{1}{2}} = 3, \text{ or } -4$$

Whence, $$2x^2+1 = 9, \text{ or } 16$$

Or, $$x^2 = 4, \text{ or } \tfrac{15}{2}$$

Extracting the square root, $$x = \pm 2, \text{ or } \pm \tfrac{1}{2}\sqrt{30}.$$

Here, we add 1 to both members, in order that the quantity without the radical may be made the same as that within. Then $(2x^2+1)$ becomes the basis of the quadratic form, instead of x.

20. Given $(x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40$, to find the values of x.

OPERATION.

$$(x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40$$

Assuming $(x-5)^{\frac{3}{2}} = y$, we have

$$y^2 - 3y = 40$$

Whence, $$y = 8, \text{ or } -5$$

But $(x-5)^{\frac{3}{2}}=y$, therefore $(x-5)^{\frac{3}{2}} = 8$, or -5

Whence, $x - 5 = 4$, or $(-5)^{\frac{2}{3}}$

Or, $x = 9$, or $5 + (-5)^{\frac{2}{3}}$

Or, $x = 9$, or $5 + \sqrt[3]{25}$.

The solution may also be carried through, as in the last example, without any change of letters.

21. Given $x^4 - 12x^3 + 44x^2 - 48x = 9009$, to find the values of x.

OPERATION.

Taking the square root,

$$\begin{array}{r|l} x^4 - 12x^3 + 44x^2 - 48x & x^2 - 6x = \text{Root.} \\ \underline{x^4} & \\ 2x^2 - 6x \;\big|\; -12x^3 + 44x^2 & \\ \quad\big|\; \underline{-12x^3 + 36x^2} & \\ \hline 8x^2 - 48x & = \text{Remainder,} \end{array}$$

which, factored, is $8(x^2 - 6x)$ Now, since the square of the root, plus the remainder, will equal the quantity whose root we have extracted, the given equation may take the form

$$(x^2 - 6x)^2 + 8(x^2 - 6x) = 9009$$

Completing the square,

$$(x^2 - 6x)^2 + 8(x^2 - 6x) + 16 = 9025$$

Extracting the square root,

$$x^2 - 6x + 4 = \pm 95$$

Or, $x^2 - 6x = 91$, or -99

Whence $x = 13$, or -7, or $3 \pm 3\sqrt{-10}$.

22. Given $x + 16 - 7\sqrt{x+16} = 10 - 4\sqrt{x+16}$, to find the values of x. Ans. $x = 9$, or -12.

23. Given $9x + \sqrt{16x^2 + 36x^3} = 15x^2 - 4$, to find the values of x. Ans. $x = \frac{4}{3}$, or $-\frac{1}{3}$, or $\frac{9 \pm \sqrt{481}}{50}$.

24. Given $x^2(x^2-4)^{-2}+\frac{6}{x^2-4}=\frac{351\,x^{-2}}{25}$, to find the values of x. Ans. $x=\pm 3$, or $\pm\frac{1}{11}\sqrt{429}$.

25. Given $x^4+2ax^3+5a^2x^2+4a^3x=d$, to find the values of x.

Ans. $x=-\frac{a}{2}\pm\frac{1}{2}\sqrt{-7a^2\pm 4\sqrt{4a^4+d}}$.

26. Given $x^3-8x^2+19x-12=0$, to find the values of x. Ans. $x=1$, 3, or 4.

Here, make the highest exponent of the unknown quantity an even number, by multiplying the equation by x.

27. Given $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$, to find the values of x.

Ans. $x=3$, or $-\frac{3}{2}$, or $\frac{-3\pm\sqrt{-55}}{4}$.

28. Given $\sqrt[m]{(1+x)^2}-\sqrt[m]{1-x^2}=\sqrt[m]{(1-x)^2}$, to find the values of x. Ans. $x=\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}$.

Here, by dividing both members by $\sqrt[m]{(1-x)^2}$, the equation takes the quadratic form.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

285. The most general form of an equation of the second degree containing two unknown quantities, is

$$x^2+axy+by^2+cx+dy+e=0,$$

which contains the squares of the unknown quantities, their product, and first powers, with a known term; and where the coefficients a, b, c, etc., represent any quantity, positive or negative, integral or fractional.

286. Two quadratic equations containing two unknown quantities will generally produce, by elimination, an equation of the fourth degree containing one unknown quantity. The rules for quadratics are not, therefore, sufficient to solve *all* simultaneous equations of the second degree.

In several cases, however, their solution may be effected by means of the ordinary rules.

CASE I.

287. When one equation is of the first degree, and the other of the second.

Equations of this class can always be solved.

Examples.

1. Given $x^2 + 3xy - y^2 = 23$, and $x + 2y = 7$, to find the values of x and y.

OPERATION.

$$x^2 + 3xy - y^2 = 23 \quad (1)$$
$$x + 2y = 7 \quad (2)$$

From (2), $x = 7 - 2y$ (3)

Substituting in (1),

$$(7 - 2y)^2 + 3y(7 - 2y) - y^2 = 23$$

Or, $49 - 28y + 4y^2 + 21y - 6y^2 - y^2 = 23$

Uniting terms, $3y^2 + 7y = 26$

Whence, $y = 2$, or $-4\frac{1}{3}$

Substituting in (3), $x = 3$, or $15\frac{2}{3}$.

Here, we find an expression for the value of one of the unknown quantities in the simple equation, which we substitute for that quantity in the other equation, and have a quadratic containing only one unknown quantity, by means of which the values of the unknown quantities are readily obtained.

On verifying the values of x and y, it will be seen that they must be taken in the same order; that is, when $x = 3$, $y = 2$; and when $x = 15\frac{2}{3}$, $y = -4\frac{1}{3}$.

2. Given $2x^2 + xy - 5y^2 = 20$, and $2x - 3y = 1$, to find the values of x and y.

Ans. $\begin{cases} x = 5, \text{ or } -9\frac{1}{4}. \\ y = 3, \text{ or } -6\frac{1}{2}. \end{cases}$

3. Given $x + \frac{1}{3} = \frac{2x+y}{3}$, and $\frac{x+y}{x} = \frac{4x-y}{2}$, to find the values of x and y.

Ans. $\begin{cases} x = 2, \text{ or } -\frac{1}{3}. \\ y = 3, \text{ or } \frac{2}{3}. \end{cases}$

CASE II.

288. When both equations are homogeneous, and of the second degree.

Equations of this class can always be solved.

Examples.

1. Given $2y^2 - 4xy + 3x^2 = 17$, and $y^2 - x^2 = 16$, to find the values of x and y.

OPERATION.

$$2y^2 - 4xy + 3x^2 = 17 \quad (1)$$
$$y^2 - x^2 = 16 \quad (2)$$

Let $y = vx$, and substituting in (1) and (2), we have

$$x^2(2v^2 - 4v + 3) = 17 \quad (3)$$
$$x^2(v^2 - 1) = 16 \quad (4)$$

From (3), $$x^2 = \frac{17}{2v^2 - 4v + 3} \quad (5)$$

From (4), $$x^2 = \frac{16}{v^2 - 1} \quad (6)$$

Equating (5) and (6), $$\frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1}$$

Clearing of fractions, $17v^2 - 17 = 32v^2 - 64v + 48$

Transposing and uniting,

$$-15v^2 + 64v = 65$$

Dividing by -15, $v^2 - \frac{64}{15}v = -\frac{65}{15}$

Whence, $v = \frac{13}{5}$, or $\frac{5}{3}$

Substituting in (6), and reducing, $x^2 = \frac{225}{9}$, or 9

Whence, $x = \pm\frac{5}{3}$, or ± 3

But $y = vx$, therefore $y = \pm\frac{13}{3}$, or ± 5.

Here it is convenient to employ an *auxiliary* quantity, and to substitute for one of the unknown quantities the product of the other by that auxiliary.

2. Given $3y^2 - x^2 = 39$, and $x^2 + 4xy = 256 - 4y^2$, to find the values of x and y.

Ans. $\begin{cases} x = \pm 6, \text{ or } \pm 102. \\ y = \pm 5, \text{ or } \mp 59. \end{cases}$

3. Given $2x^2 + 3xy = 26$, and $3y^2 + 2xy = 39$, to find the values of x and y.

Ans. $\begin{cases} x = \pm 2. \\ y = \pm 3. \end{cases}$

4. Given $x^2 + xy + 4y^2 = 6$, and $3x^2 + 8y^2 = 14$, to find the values of x and y.

Ans. $\begin{cases} x = \pm 2, \text{ or } \pm \frac{1}{5}\sqrt{10}. \\ y = \pm 1, \text{ or } \mp \frac{2}{5}\sqrt{10}. \end{cases}$

CASE III.

289. When the given equations are symmetrical.

Quadratic and higher equations of this class are frequently readily solved by taking advantage of the relations existing between the sum, difference, and product of two quantities.

It is often convenient, also, to employ auxiliary unknown quantities.

EXAMPLES.

1. Given $x^2 + y^2 = 25$, and $xy = 12$, to find the values of x and y.

OPERATION.

$$x^2 + y^2 = 25 \quad (1)$$
$$xy = 12 \quad (2)$$

Multiplying (2) by 2, $2xy = 24$ (3)

Adding (3) to (1), $x^2 + 2xy + y^2 = 49$ (4)

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 1$ (5)

Extracting square root of (4), $x + y = \pm 7$ (6)

Extracting square root of (5), $x - y = \pm 1$ (7)

Adding (7) to (6), $2x = \pm 8$, or ± 6

Subtracting (7) from (6), $2y = \pm 6$, or ± 8

Whence, $x = \pm 4$, or ± 3

Also, $y = \pm 3$, or ± 4.

It is evident that this example might be classed under Case II. as well as under Case III.; but the method here adopted gives a simple solution.

It must not be inferred from the result, that x and y are equal to each other; for when $x = 4$, $y = 3$, and when $x = 3$, $y = 4$. Whenever two simultaneous equations are symmetrical in their *signs*, as well as in other respects, the letters, obviously, may be exchanged without affecting the equation; consequently the values of the letters must be interchangeable, and when the two values of one letter are found, the same values may be assigned to the other letter, the order being reversed.

2. Given $x^2 + y^2 = 208$, and $x + y = 20$, to find the values of x and y.

OPERATION.

$$x^2 + y^2 = 208 \quad (1)$$
$$x + y = 20 \quad (2)$$

Multiplying (1) by 2,	$2x^2 + 2y^2 = 416$	(3)
Squaring (2),	$x^2 + 2xy + y^2 = 400$	(4)
Subtracting (4) from (3),	$x^2 - 2xy + y^2 = 16$	(5)
Extracting square root,	$x - y = \pm 4$	(6)
Equation (2),	$x + y = 20$	(7)
Adding (6) to (7),	$2x = 24$, or 16	
Subtracting (6) from (7),	$2y = 16$, or 24	
Whence,	$x = 12$, or 8	
Also,	$y = 8$, or 12.	

3. Given $x^3 - y^3 = 19$, and $x^2y - xy^2 = 6$, to find the values of x and y.

OPERATION.

	$x^3 - y^3 = 19$	(1)
	$x^2y - xy^2 = 6$	(2)
Multiplying (2) by 3,	$3x^2y - 3xy^2 = 18$	(3)
Subtracting (3) from (1),	$x^3 - 3x^2y + 3xy^2 - y^3 = 1$	(4)
Extracting cube root of (4),	$x - y = 1$	(5)
Dividing (2) by (5),	$xy = 6$	(6)
Squaring (5),	$x^2 - 2xy + y^2 = 1$	(7)
Multiplying (6) by 4,	$4xy = 24$	(8)
Adding (7) and (8),	$x^2 + 2xy + y^2 = 25$	(9)
Extracting square root of (9),	$x + y = \pm 5$	(10)
Equation (5),	$x - y = 1$	
Adding (5) to (10)	$2x = 6$, or -4	
Subtracting (5) from (10),	$2y = 4$, or -6	
Whence,	$x = 3$, or -2	
Also,	$y = 2$, or -3.	

Here, as the original equations are not symmetrical in their *signs*, the values of x and y are not interchangeable.

4. Given $\frac{x^2}{y}+\frac{y^2}{x}=18$, and $x+y=12$, to find the values of x and y.

OPERATION.

$$\frac{x^2}{y}+\frac{y^2}{x}=18 \quad (1)$$

$$x+y=12 \quad (2)$$

Let $x=u+v$ (3)

Also, $y=u-v$ (4)

Adding (3) and (4), $x+y=2u=12$

Whence, $u=6$

Clearing (1) of fractions, $x^3+y^3=18xy$

By substitution, $(6+v)^3+(6-v)^3=18(6+v)(6-v)$

Reducing, $432+36v^2=648-18v^2$

Or, $54v^2=216$

Whence, $v=\pm 2$

Substituting in (3), $x=8$, or 4

Substituting in (4), $y=4$, or 8.

Here, we substitute for the unknown quantities, the *sum* and *difference* of *two auxiliary* quantities, as a convenient device to facilitate the solution.

5. Given $x^3+y^3=35$, and $x+y=5$, to find the values of x and y.

Ans. $\begin{cases} x=3, \text{ or } 2. \\ y=2, \text{ or } 3. \end{cases}$

6. Given $x^2+y^2=20$, and $x^2-y^2=12$, to find the values of x and y.

Ans. $\begin{cases} x=\pm 4. \\ y=\pm 2. \end{cases}$

7. Given $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y=19$, and $x^2+xy+y^2=133$, to find the values of x and y.

Ans. $\begin{cases} x=9, \text{ or } 4. \\ y=4, \text{ or } 9. \end{cases}$

Divide the second equation by the first, and to the result add the first.

8. Given $x+y=9$, and $\sqrt[3]{x}+\sqrt[3]{y}=3$, to find the values of x and y.

$$\text{Ans.} \begin{cases} x=8, \text{ or } 1. \\ y=1, \text{ or } 8. \end{cases}$$

Substituting v for $\sqrt[3]{x}$, and z for $\sqrt[3]{y}$, the equations will become $v^3+z^3=9$, and $v+z=3$.

290. Sometimes one of the given equations, or some combination of the two equations, takes the quadratic form.

9. Given $x^2+y^2+x+y=18$, and $xy=6$, to find the values of x and y.

$$\text{Ans.} \begin{cases} x=3, \text{ or } 2, \text{ or } -3\pm\sqrt{3}. \\ y=2, \text{ or } 3, \text{ or } -3\mp\sqrt{3}. \end{cases}$$

Adding twice the second equation to the first, the sum may be put under the form

$$(x+y)^2+(x+y)=30,$$

from which, by completing the square, we obtain

$$x+y=5, \text{ or } -6.$$

The signs $\pm$ and $\mp$ being taken in connection in the above answer, the one is to be understood as the reverse of the other. That is, x and y must always take opposite signs.

10. Given $x^2+y^2-x-y=18$, and $xy+x+y=19$, to find the values of x and y.

$$\text{Ans.} \begin{cases} x=4, \text{ or } 3, \text{ or } -4\pm\sqrt{-11}. \\ y=3, \text{ or } 4, \text{ or } -4\mp\sqrt{-11}. \end{cases}$$

11. Given $x^2+3x+y=73-2xy$, and $y^2+3y+x=44$, to find the values of x and y.

$$\text{Ans.} \begin{cases} x=4, \text{ or } 16, \text{ or } -12\pm\sqrt{58}. \\ y=5, \text{ or } -7, \text{ or } -1\mp\sqrt{58}. \end{cases}$$

291. The following equations may be solved by such of the preceding methods as may be deemed most convenient, or by such devices as skill and ingenuity may suggest.

EXAMPLES.

1. Given $\left\{\begin{matrix} x-y=3 \\ x^2+y^2=117 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=9, \text{ or } -6. \\ y=6, \text{ or } -9. \end{matrix}\right.$

2. Given $\left\{\begin{matrix} 2(x-y)=11 \\ xy=20 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=8, \text{ or } -2\frac{1}{2}. \\ y=2\frac{1}{2}, \text{ or } -8. \end{matrix}\right.$

3. Given $\left\{\begin{matrix} x-y=2 \\ x^3-y^3=98 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=5, \text{ or } -3. \\ y=3, \text{ or } -5. \end{matrix}\right.$

4. Given $\left\{\begin{matrix} 10x+y=3xy \\ y-x=2 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=2, \text{ or } -\frac{1}{3}. \\ y=4, \text{ or } \frac{5}{3}. \end{matrix}\right.$

5. Given $\left\{\begin{matrix} x^2+xy=84 \\ xy+y^2=60 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=\pm 7. \\ y=\pm 5. \end{matrix}\right.$

6. Given $\left\{\begin{matrix} \sqrt{x^2+y^2}=10 \\ x^2-y^2=28 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=\pm 8. \\ y=\pm 6. \end{matrix}\right.$

7. Given $\left\{\begin{matrix} x^2y+xy^2=30 \\ x^4y^2+x^2y^4=468 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=3, \text{ or } 2. \\ y=2, \text{ or } 3. \end{matrix}\right.$

8. Given $\left\{\begin{matrix} x^2+y^2=25 \\ 2xy=24 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=\pm 4, \text{ or } \pm 3. \\ y=\pm 3, \text{ or } \pm 4. \end{matrix}\right.$

9. Given $\left\{\begin{matrix} \frac{1}{x}+\frac{1}{y}=\frac{1}{2} \\ x^2y+xy^2=162 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x=6, \text{ or } 3, \text{ or } \frac{3}{2}(-3\pm\sqrt{17}). \\ y=3, \text{ or } 6, \text{ or } \frac{3}{2}(-3\mp\sqrt{17}). \end{matrix}\right.$

10. Given $\left\{\begin{matrix} \frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{10}{3} \\ x^2+y^2=45 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x=\pm 6. \\ y=\pm 3. \end{matrix}\right.$

11. Given $\left\{\begin{matrix} x^2+y^2=\frac{5}{2}xy \\ x-y=\frac{1}{2}xy \end{matrix}\right\}$. Ans. $\begin{cases} x=4, \text{ or } -2. \\ y=2, \text{ or } -4. \end{cases}$

12. Given $\left\{\begin{matrix} x+y=a \\ x^2+y^2=b \end{matrix}\right\}$. Ans. $\begin{cases} x=\frac{a}{2}\pm\frac{\sqrt{2b-a^2}}{2}. \\ y=\frac{a}{2}\mp\frac{\sqrt{2b-a^2}}{2}. \end{cases}$

13. Given $\left\{\begin{matrix} 3x^2-2xy=15 \\ 2x+3y=12 \end{matrix}\right\}$.

Ans. $\begin{cases} x=3, \text{ or } -1\frac{2}{13}. \\ y=2, \text{ or } 4\frac{10}{13}. \end{cases}$

14. Given $\left\{\begin{matrix} 6x^2+2y^2=5xy+12 \\ 2xy+3x^2=3y^2-3 \end{matrix}\right\}$.

Ans. $\begin{cases} x=\pm 2, \text{ or } \pm\frac{5}{31}\sqrt{31}. \\ y=\pm 3, \text{ or } \mp\frac{6}{31}\sqrt{31}. \end{cases}$

15. Given $\left\{\begin{matrix} xz=y^2 \\ (x+y)(z-x-y)=3 \\ (x+y+z)(z-x-y)=7 \end{matrix}\right\}$.

Ans. $\begin{cases} x=\pm 1, \text{ or } \pm 9. \\ y=\pm 2, \text{ or } \mp 6. \\ z=\pm 4. \end{cases}$

PROBLEMS

LEADING TO QUADRATIC EQUATIONS.

1. A says to B, "The sum of our money is \$18"; B replies, "But if twice the number of your dollars were multiplied by mine, the product would be \$154." How many dollars had each?

Ans. A, \$7; B, \$11.

2. The difference of two numbers is 5, and the sum of their squares is 193. What are those numbers?

Ans. 12 and 7.

3. A and B have each a small field, in the shape of an exact square, and it requires 200 rods of fence to enclose both. The contents of these fields are 1300 square rods. What is the value of each, at $2.25 per square rod? Ans. One, $900; other, $2025.

4. There are two numbers whose product is 77, and the difference of whose squares is to the square of their difference as 9 to 2. Required the numbers.

Ans. 11 and 7.

5. Two gentlemen, A and B, were speaking of their ages. A said that the product of their ages was 750; B replied, that if his age were increased 7 years, and A's were lessened 2 years, their product would be 851. Required their ages. Ans. A's, 25 years; B's, 30 years.

6. John Smith's garden is a rectangle, and contains 15,000 square yards, exclusive of a walk, 7 yards wide, which surrounds it, and contains 3696 square yards. Required the length and breadth of the garden.

Ans. Length, 150 yards; breadth, 100 yards.

7. What two numbers are those whose difference multiplied by the less produces 42, and by their sum 133?

Ans. 13 and 6.

8. A man bought 10 ducks and 12 turkeys for $22.50. He bought 4 more ducks for $6 than turkeys for $5. What was the price of each?

Ans. Of a duck, $0.75; of a turkey, $1.25.

9. A man purchased a farm in the form of a rectangle, whose length was four times its breadth. It cost $\frac{1}{4}$ as many dollars per acre as the field was rods in length, and the number of dollars paid for the farm was four times the

number of rods round it. Required the price of the farm, and its length and breadth.

Ans. Price, $1600; length 160 rods, breadth 40 rods.

10. I have two cubic blocks of marble, whose united lengths are 20 inches, and contents 2240 cubic inches. Required the surface of each.

Ans. 864 square inches; 384 square inches.

11. A's and B's shares in a speculation altogether amount to $500; they sell out at *par*, A at the end of 2 years, B of 8, and each receives in capital and profits $297. How much did each embark?

Ans. A, $275; B, $225.

12. A farmer bought as many sheep as cost him $300; after reserving 15 out of the number, he sold the remainder for $270, and gained $0.50 per head on them. How many sheep did he buy? Ans. 75.

13. A person has $1300, which he divides into two portions, and loans at different rates of interest, so that the two portions produce equal returns. If the first portion had been loaned at the second rate of interest, it would have produced $36, and if the second portion had been loaned at the first rate of interest, it would have produced $49. Required the rates of interest.

Ans. 7 and 6 per cent.

14. Two men, A and B, bought a farm of 200 acres, for which they paid $200 each. On dividing the land, A says to B, "If you will let me have my part in the situation which I shall choose, you shall have so much more land than I that mine shall cost 75 cents per acre more than yours." B accepted the proposal. How much land did each have, and what was the price of each per acre?

Ans. A, 81.867 acres, at $2.443; B, 118.133 acres, at $1.693.

15. A and B start at the same time from two distant towns. At the end of 7 days, A is nearer to the half-way house than B is, by 5 miles more than A's day's journey. At the end of 10 days they have passed the half-way house, and are distant from each other 100 miles. Now it will take B three days longer to perform the whole journey than it will A. Required the distance of the towns, and the rate of walking of A and B.

Ans. Distance of the towns, 450 miles; A's rate, 30 miles, and B's rate, 25 miles, per day.

THEORY OF QUADRATIC EQUATIONS.

292. Every complete quadratic equation, it has been shown (Art. 274), may be reduced to the form

$$x^2 + px = q,$$

where p and q represent known quantities, positive or negative, integral or fractional. This form may also represent a pure quadratic, by considering $p = 0$.

The values of x in this equation are

$$-\frac{p}{2} + \sqrt{q + \frac{p^2}{4}},$$

$$-\frac{p}{2} - \sqrt{q + \frac{p^2}{4}};$$

and, since no other values necessarily result from the general equation, we infer that

Every equation of the second degree has TWO ROOTS, *and only two.*

293. If we add the two roots of the general equation (Art. 292), their sum is $-p$, and if we multiply them together, their product is

$$\frac{p^2}{4} - \left(q + \frac{p^2}{4}\right) = -q.$$

That is,

1. *The algebraic* SUM *of the two roots of a quadratic equation is equal to the coefficient of the second term, with its sign changed.*

2. *The* PRODUCT *of the two roots is equal to the second member, with its sign changed.*

294. Representing by r and r' the two roots of the quadratic equation

$$x^2 + px - q = 0,$$

we have $$x = r, \text{ or } x = r',$$

that is, $$x - r = 0, \text{ or } x - r' = 0.$$

Multiplying the last two expressions together, we have

$$(x - r)(x - r') = 0,$$

$$x^2 - (r + r')x + rr' = 0.$$

But, by Article 293,

$$r + r' = -p, \text{ and } rr' = -q;$$

whence,

$$x^2 + px - q = (x - r)(x - r') = 0.$$

That is,

If all the terms of a quadratic equation be transposed to the first member, it may be resolved into the two binomial factors formed by subtracting each of the two roots of the equation from the unknown quantity.

295. A QUADRATIC EXPRESSION is a trinomial which contains the first and second powers of some letter or quantity.

The principle established in the last article furnishes a method of resolving any quadratic expression into simple binomial factors.

EXAMPLES.

1. Resolve $x^2 - 4x - 60$ into binomial factors.

OPERATION.

Assume $x^2 - 4x - 60 = 0$

Whence, $x = 10$, or $x = -6$

Hence, by Art. 294,

$$x^2 - 4x - 60 = (x - 10)(x + 6).$$

2. Resolve $3x^2 - 10x - 25$ into binomial factors.

OPERATION.

$$3x^2 - 10x - 25$$

Separating the factor 3, we have

$$3\left(x^2 - \frac{10x}{3} - \frac{25}{3}\right)$$

Assuming the quantity within the parenthesis equal to 0, and transposing, we obtain

$$x^2 - \frac{10x}{3} = \frac{25}{3}$$

Whence, $x = 5$, $x = -\frac{5}{3}$

And, by Art. 294,

$$3x^2 - 10x - 25 = 3(x - 5)(x + \tfrac{5}{3}).$$

3. Resolve $x^2 - 6x + 8$ into binomial factors.

Ans. $(x - 4)(x - 2)$.

4. Resolve $x^2 + 73x + 780$ into binomial factors.

Ans. $(x + 60)(x + 13)$.

5. Resolve $2x^2 + x - 6$ into binomial factors.

Ans. $2(x + 2)(x - \frac{3}{2})$.

6. Resolve $x^2 + \frac{52}{3}x - \frac{256}{3}$ into binomial factors.

Ans. $(x - 4)(x + \frac{64}{3})$.

FORMATION OF EQUATIONS.

296. The principles established in Articles 293 and 294 furnish methods of readily forming a quadratic equation when its roots are given.

EXAMPLES.

1. Required the equation whose roots are 4 and $-\frac{7}{4}$.

OPERATION.

$$(x-4)\,(x+\tfrac{7}{4}) = x^2 - \frac{9x}{4} - 7 = 0$$

Or, clearing of fractions, $4x^2 - 9x - 28 = 0$

Whence, $4x^2 - 9x = 28.$

2. Form the quadratic equation whose roots are 1 and -2. Ans. $x^2 + x = 2.$

3. Form the quadratic equation whose roots are 4 and 5. Ans. $x^2 - 9x = -20.$

4. Form the quadratic equation whose roots are $1+\sqrt{5}$ and $1-\sqrt{5}$. Ans. $x^2 - 2x = 4.$

5. Find the quadratic equation whose roots are 3 and $-\frac{3}{5}$. Ans. $5x^2 - 12x = 9.$

6. Find the quadratic equation whose roots are 7 and $-6\frac{1}{3}$. Ans. $x^2 - \frac{2x}{3} = 44\frac{1}{3}.$

7. Find the quadratic equation whose roots are $m+\sqrt{n}$ and $m-\sqrt{n}$. Ans. $x^2 - 2mx = n - m^2.$

DISCUSSION OF THE GENERAL EQUATION.

297. The values of p and q in the general equation may be either positive or negative. If those quantities be considered essentially positive, and the signs be expressed, we shall have *four forms.*

$$x^2 + px = q, \qquad x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}; \qquad (1)$$

$$x^2 - px = q, \qquad x = \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}; \qquad (2)$$

$$x^2 + px = -q, \qquad x = -\frac{p}{2} \pm \sqrt{-q + \frac{p^2}{4}}; \qquad (3)$$

$$x^2 - px = -q, \qquad x = \frac{p}{2} \pm \sqrt{-q + \frac{p^2}{4}}. \qquad (4)$$

298. Since, in the general equation, q is positive in the first and second forms, and negative in the third and fourth, the roots must have different signs in the first two forms, and the same sign in the last two (Art. 293).

Moreover, since $\frac{p}{2}$ must be numerically greater than $\sqrt{-q + \frac{p^2}{4}}$, the signs of the roots in the last two forms must be controlled by the sign of $\frac{p}{2}$. Hence,

1. *In the first and second forms, one root is positive and the other negative.*

2. *In the third form, both roots are negative; and in the fourth, both roots are positive.*

299. It is evident that the quantities under the radical sign of the roots in the first two forms can never be negative, and that they can be negative in the last two forms only when $\frac{p^2}{4}$ is numerically less than q. Hence,

1. *In the first and second forms, both roots are always real.*

2. *In the third and fourth forms, both roots are imaginary when the square of half the coefficient of* x *is numerically less than the second member; otherwise they are real.*

300. It is evident that the radical portion of the roots can never become 0 in the first two forms; but if $\frac{p^2}{4} = q$, the radical portion will become 0 in the last two forms, and both roots will be $-\frac{p}{2}$, or $\frac{p}{2}$. Hence,

1. *In the first and second forms, the two roots are always numerically unequal.*

2. *In the third and fourth forms, the two roots are equal when the square of half the coefficient of* x *is numerically equal to the second member; otherwise they are unequal.*

DISCUSSION OF PROBLEMS

LEADING TO QUADRATIC EQUATIONS.

301. In the discussion of problems leading to quadratic equations, we find involved the same general principles which have been established in connection with simple equations (Arts. 173–186), but with certain peculiarities.

These peculiarities will here be considered. They arise from two facts: —

1. That every quadratic equation has *two roots;* and
2. That these roots are sometimes *imaginary.*

302. In the solution of problems involving quadratics, it has been observed that the positive root of the equation is usually the true answer; and that, when both roots are positive, there may be two answers, either of which conforms to the given conditions.

The reason why results are sometimes obtained which do not apply to the given problem, and therefore are not admissible, is that the algebraic mode of expression is more general than ordinary language, and thus the equation which conforms properly to the conditions of the problem, will also apply to other conditions.

1. Find a number such, that twice its square added to three times the number, may be 65.

OPERATION.

Let x = the number. (1)

Then $2x^2 + 3x = 65$ (2)

Or, $x^2 + \frac{3x}{2} = \frac{65}{2}$

Completing the square, $x^2 + \frac{3x}{2} + \frac{9}{16} = \frac{529}{16}$

Or, $x + \frac{3}{4} = \pm \frac{23}{4}$

Whence, $x = 5$, or $-\frac{13}{2}$.

The positive value alone gives a solution to the problem in the sense in which it is proposed.

To interpret the negative value, observe, that if we change x into $-x$, in equation (2), the term $3x$, only, changes its sign, which requires it to be subtracted from $2x^2$ to produce 65. In this new enunciation the values of x will be $\frac{13}{2}$, or -5, which values only differ from the others in their signs. We therefore may consider the negative solution, $-\frac{13}{2}$, taken independently of its sign, the proper answer to the analogous problem (Art. 181):—

Find a number such, that twice its square *diminished by* three times the number, may be 65.

2. A farmer bought some sheep for \$72, and found that if he had bought 6 more for the same money, he would have paid \$1 less for each. How many sheep did he buy?

OPERATION.

Let x = the number of sheep bought.

Then $\frac{72}{x}$ = the price paid for one,

and $\frac{72}{x+6}$ = the price, if 6 more.

Therefore,	$\frac{72}{x} = \frac{72}{x+6} + 1$
Or,	$72\,(x+6) = 72x + x\,(x+6)$
Whence,	$x^2 + 6x = 432$
Completing the square,	$x^2 + 6x + 9 = 441$
Or,	$x + 3 = \pm 21$
Whence,	$x = 18$, or -24.

Here the negative result is not admissible as a solution of the problem in the present form; the number of sheep, therefore, was 18.

If in the given problem "6 more" be changed to 6 *fewer*, and "\$1 less" to \$1 *more*, 24 will be the true answer.

Hence, we infer that

A negative result, obtained as one of the answers to a problem, is sometimes the answer to another analogous problem, formed by attributing to the unknown quantity a quality directly opposite to that which has been attributed to it.

INTERPRETATION OF IMAGINARY RESULTS.

303. It has been shown (Art. 299) under what circumstances a quadratic equation will be in form to produce imaginary roots. It is now proposed to interpret such results.

Let it be required to divide 10 into two such parts that their product shall be 26.

OPERATION.

Let	$x =$ one of the parts.
Then	$10 - x =$ the other.
Therefore,	$x\,(10 - x) = 26$
Or,	$10x - x^2 = 26$
Changing signs,	$x^2 - 10x = -26$

Completing the square, $x^2 - 10x + 25 = -1$

Or, $x - 5 = \pm\sqrt{-1}$

Whence, $x = 5 \pm \sqrt{-1}$

Thus, we obtain an imaginary result. We therefore conclude that the problem cannot be solved numerically; in fact, if we call one of the parts $5 + y$, the other must be $5 - y$, and their product will be $25 - y^2$, which, so long as y is numerical, is less than 25. But we are required to find two *numbers* whose sum is 10 and product 26; there are, then, no such *numbers.*

Had it been required to find two *expressions,* which, when combined by addition, shall give 10, and by multiplication 26, the answer $5 + \sqrt{-1}$ and $5 - \sqrt{-1}$ would have satisfied the conditions.

The given problem, however, expresses conditions incompatible with each other, and, consequently, is impossible. Hence,

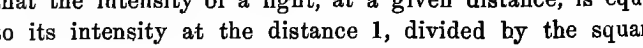
Imaginary results indicate that the problem is impossible.

PROBLEM OF THE LIGHTS.

304. The principles of interpretation will be further illustrated in the discussion of the following general

PROBLEM.

Find upon the line which joins two lights, A and B, the point which is equally illuminated by them; admitting that the intensity of a light, at a given distance, is equal to its intensity at the distance 1, divided by the square of the given distance.



Assume A as the origin of distances, and regard all distances estimated to the right as positive.

Let a denote the intensity of the light A, at the distance 1; b, the intensity of the light B, at the distance 1; and c, the distance A B, between the two lights.

Suppose C the point of equal illumination, and let x represent the distance from it to A, or the distance A C. Then $c - x$ will represent the distance B C.

By the conditions of the problem, since the intensity of the light A, at the distance 1, is a, at the distance x it is $\frac{a}{x^2}$; and since the intensity of the light B, at the distance 1, is b, at the distance $c - x$ it is $\frac{b}{(c-x)^2}$. But, by supposition, at C these intensities are equal; hence,

$$\frac{a}{x^2} = \frac{b}{(c-x)^2}; \quad \text{or} \quad \frac{(c-x)^2}{x^2} = \frac{b}{a};$$

whence,
$$\frac{c-x}{x} = \frac{\pm\sqrt{b}}{\sqrt{a}}.$$

From this equation we obtain as the values of x: —

$$x = \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right).$$

$$x = \frac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right).$$

Since both a and b are positive, the two values of x are both real. Hence,

There are two points of equal illumination on the line of the lights.

Since there are *two* lights, c must always be greater than 0, consequently neither a, b, nor c can be 0. The problem, then, admits properly of only these three different suppositions: —

1. $a > b$; 2. $a < b$; and, 3. $a = b$.

We shall now discuss the values of x, under each of these suppositions.

1. $a > b$.

In this case, the first value of x is less than c; because $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, being a proper fraction, is less than 1. This value of x is also greater than $\frac{1}{2}c$; because the denominator being less than twice the numerator, the fraction is greater than $\frac{1}{2}$. Hence, the first point of equal illumination is at C, between the two lights, but nearer the lesser one.

The second value of x is greater than c; because $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$, being an improper fraction, is greater than 1. Hence, the second point is at C′, in the prolongation of the line A B, beyond the lesser light.

These results agree with the supposition. For, if a is greater than b, then B is evidently the lesser light. Hence, both points of equal illumination will be nearer the second light, and since the two lights emit rays in all directions, one of the points must be in the prolongation of A B beyond both lights.

2. $a < b$.

In this case, the first value of x is positive. It is also less than $\frac{1}{2}c$; because $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, having the denominator greater than twice the numerator, is less than $\frac{1}{2}$. Hence, the first point of illumination is between the lights, but nearer A, the lesser light.

The second value of x is negative, because the denominator $\sqrt{a}-\sqrt{b}$ is negative; which must be interpreted as measuring distance from A towards the left (Art. 302).

Hence, the second point is at C″, in the prolongation of the line at the left of the lesser light.

These results correspond with the supposition, the case being the same as the preceding one, except that A is now the lesser light.

3. $a = b$.

In this case, the first value of x is positive, and equal to $c - x$, and both reduce to $\frac{c}{2}$. Hence, the first point of equal illumination is midway between the two lights.

The second value of x is not finite; because $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$, having $\sqrt{a} = \sqrt{b}$, reduces to $\frac{\sqrt{a}}{0} = \infty$, which indicates that no finite value can be assigned to x. Hence, there is no second point of equal illumination in the line A B, or its prolongation, at a finite distance from it.

These results agree with the supposition. For, since the lights are of equal intensity, a point of equal illumination will obviously be midway between them; and it is evident that there can be no other like point in their line.

The preceding discussion illustrates the precision with which algebraic processes will conform to every allowable interpretation of the enunciation of a problem.

RATIO AND PROPORTION.

305. The Ratio of one quantity to another of the same kind is the quotient arising from dividing the first quantity by the second (Art. 158).

Thus, the ratio of a to b is $\frac{a}{b}$, or $a : b$.

306. The TERMS of a ratio are the two quantities required to form it. Of these the first is called the *antecedent*, and the second the *consequent*.

Thus, in $a : b$, a and b are the terms, of which a is the antecedent, and b the consequent.

307. A PROPORTION is an equality of ratios.

Thus, if the ratios $a : b$ and $c : d$ are equal, they form a proportion, which may be written

$$a : b = c : d, \quad \text{or} \quad a : b :: c : d.$$

308. The TERMS of a proportion are the four terms of its two ratios. The first and third of the terms are called the *antecedents;* the second and fourth, *consequents;* the first and last, *extremes;* the second and third, *means;* and the terms of each ratio constitute a *couplet.*

Thus, in $a : b :: c : d$, a and c are antecedents; b and d, consequents; a and d, extremes; b and c, means; a and b, the first couplet; and c and d, the second couplet.

309. A PROPORTIONAL is any one of the terms of a proportion, and a MEAN PROPORTIONAL between two quantities is either of the two means, when they are the same quantity.

Thus, in the proportion $a : b :: c : d$, the fourth term, d, is the fourth proportional to a, b, and c, taken in their order; and in the proportion $a : b :: b : d$, b is a mean proportional between the quantities a and d.

310. A CONTINUED PROPORTION is one in which each consequent is the same as the next antecedent; as, the proportion

$$a : b :: b : c :: c : d :: d : e.$$

PROPERTIES OF PROPORTIONS.

311. *When four quantities are proportional, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be the four quantities; then, since they are proportional (Art. 307),

$$\frac{a}{b} = \frac{c}{d};$$

whence, $$ad = bc.$$

Hence, *if three quantities be in continued proportion, the product of the two extremes is equal to the square of the mean.* Thus, if

$$a : b :: b : c,$$

then $$ac = bb = b^2.$$

Likewise, *if three terms of a proportion be given, the fourth may be found.* Thus, if

$$a : b :: c : x$$

Then $$ax = bc$$

Whence, $$x = \frac{bc}{a}.$$

312. *If the product of two quantities be equal to the product of two others, two of them may be made the extremes and the other two the means of a proportion.*

Let $ad = bc$; dividing by bd, we have $\frac{a}{b} = \frac{c}{d}$;

whence, $$a : b :: c : d.$$

313. *If four quantities be in proportion, they will be in proportion by* ALTERNATION; *that is, the antecedents will have the same ratio to each other that the consequents have.*

If $a : b :: c : d$, then $a : c :: b : d$.

For, $\frac{a}{b} = \frac{c}{d}$; multiplying by $\frac{b}{c}$, we have $\frac{a}{c} = \frac{b}{d}$;

whence, $a : c :: b : d$.

314. *If four quantities be in proportion, they will be in proportion by* INVERSION ; *that is, the second term will have the same ratio to the first that the fourth has to the third.*

If $a : b :: c : d$, then $b : a :: d : c$.

For, $\frac{a}{b} = \frac{c}{d}$; dividing 1 by each member, we have

$$\frac{b}{a} = \frac{d}{c};$$

whence, $b : a :: d : c$.

315. *If four quantities be in proportion, they will be in proportion by* COMPOSITION ; *that is, the sum of the first and second terms will have to the second the same ratio that the sum of the third and fourth has to the fourth.*

If $a : b :: c : d$, then $a + b : b :: c + d : d$.

For, $\frac{a}{b} = \frac{c}{d}$; adding 1 to both members, we have

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ or } \frac{a+b}{b} = \frac{c+d}{d};$$

whence, $a + b : b :: c + d : d$.

316. *If four quantities be in proportion, they will be in proportion by* DIVISION ; *that is, the difference of the first and second terms will have to the second the same ratio that the difference of the third and fourth has to the fourth.*

If $a : b :: c : d$, then $a - b : b :: c - d : d$.

For, $\frac{a}{b} = \frac{c}{d}$; subtracting 1 from each member, we have

$$\frac{a}{b} - 1 = \frac{c}{d} - 1, \text{ or } \frac{a-b}{b} = \frac{c-d}{d};$$

whence, $a - b : b :: c - d : d.$

317. *If four quantities be in proportion, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

If $a : b :: c : d$, then $a + b : a - b :: c + d : c - d.$

By Art. 315, $\frac{a+b}{b} = \frac{c+d}{d}$,

and, by Art. 316, $\frac{a-b}{b} = \frac{c-d}{d}$; therefore,

$$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}, \text{ or } \frac{a+b}{a-b} = \frac{c+d}{c-d};$$

whence, $a + b : a - b :: c + d : c - d.$

318. *Quantities which are proportional to the same quantities are proportional to each other.*

If $a : b :: e : f$ and $c : d :: e : f$, then $a : b :: c : d.$

For, $\frac{a}{b} = \frac{e}{f}$, and $\frac{c}{d} = \frac{e}{f}$; therefore $\frac{a}{b} = \frac{c}{d}$;

whence, $a : b :: c : d.$

319. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

If $a : b :: c : d :: e : f$, then $a : b :: a + c + e : b + d + f.$

For, by Art. 311,

$$ad = bc, \text{ and } af = be; \text{ also, } ab = ba.$$

Adding, $ab + ad + af = ba + bc + be,$

or, $a(b + d + f) = b(a + c + e);$

whence, by Art. 312,

$$a : b :: a + c + e : b + d + f.$$

320. *When four quantities are in proportion, if the first and second be multiplied or divided by any quantity, as also the third and fourth, the resulting quantities will be proportional.*

If $a : b :: c : d$, then $ma : mb :: nc : nd$.

For, $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{ma}{mb} = \frac{nc}{nd}$;

whence, $ma : mb :: nc : nd$.

In like manner, $\frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$.

Either m or n may be made equal to unity; that is, either couplet may be multiplied or divided without multiplying or dividing the other.

321. *When four quantities are in proportion, if the first and third be multiplied or divided by any quantity, as also the second and fourth, the resulting quantities will be proportional.*

If $a : b :: c : d$, then $ma : nb :: mc : nd$.

For, $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{ma}{b} = \frac{mc}{d}$, and $\frac{ma}{nb} = \frac{mc}{nd}$;

whence, $ma : nb :: mc : nd$.

In like manner, $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$.

Either m or n may be made equal to unity.

322. *If there be two sets of proportional quantities, the products of the corresponding terms will be proportional.*

If $a : b :: c : d$ and $e : f :: g : h$,

then $ae : bf :: cg : dh$.

For, $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{h}$; therefore $\frac{ae}{bf} = \frac{cg}{dh}$;

whence, $ae : bf :: cg : dh$.

323. *If four quantities be in proportion, like powers or roots of these quantities will be proportional.*

If $a : b :: c : d$, then $a^n : b^n :: c^n : d^n$.

For, $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a^n}{b^n} = \frac{c^n}{d^n}$ and $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$, where n may be whole or fractional; whence,

$$a^n : b^n :: c^n : d^n, \text{ and } a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

324. *If three quantities be in continued proportion, the first is to the third as the square of the first is to the square of the second.*

If $a : b :: b : c$, then $a : c :: a^2 : b^2$.

For, $\frac{a}{b} = \frac{b}{c}$; multiplying by $\frac{a}{b}$, we have $\frac{a^2}{b^2} = \frac{a}{c}$;

whence, $a : c :: a^2 : b^2$.

In like manner we may show that

If $a : b :: b : c :: c : d$, then $a : d :: a^3 : b^3$,

where the four quantities, a, b, c, d, are in continued proportion.

NOTE. — The ratio $a^2 : b^2$ is called the *duplicate ratio*, and the ratio $a^3 : b^3$ the *triplicate ratio*, of a to b.

PROBLEMS IN PROPORTION.

1. The last three terms of a proportion being 18, 6, and 27, what is the first term? Ans. 4.

2. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. Ans. $\frac{1}{6}$.

3. Find a mean proportional to 2 and 8. Ans. 4.

4. Find two numbers in the ratio of $2\frac{1}{2}$ to 2, such that, when each is diminished by 5, they shall be in the ratio of $1\frac{1}{3}$ to 1. Ans. 25 and 20.

5. Divide 50 into two such parts, that the greater, increased by 3, shall be to the less, diminished by 3, as 3 to 2. Ans. 27 and 23.

6. Each of two vessels contains a mixture of wine and water; a mixture consisting of equal measures from the two vessels contains as much wine as water; and another mixture, consisting of four measures from the first vessel and one from the second, is composed of wine and water in the ratio of 2 to 3. Find the ratio of wine to water in each of the vessels.

Ans. In the first, 1 : 2; in the second, 2 : 1.

7. If the increase in the number of male and female criminals be 1.8 per cent., while the decrease in the number of males alone is 4.6 per cent., and the increase in the number of females is 9.8 per cent.; compare the number of male and female criminals, respectively, at the first time mentioned. Ans. 4 females to 5 males.

8. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds, whereas, if it had been moving in the same direction with him, it would have passed him in 30 seconds. Compare the rates of the two trains. Ans. 8 : 7.

VARIATION.

325. Variation, or general proportion, is an abridged method of expressing common proportion.

Thus, if A and B be two sums of money, loaned for equal times, at the same rate of interest, then

A : B :: A's interest : B's interest,

or, in an abridged form, by expressing only *two* terms, the interest varies as the principal; thus (Art. 25),

The interest $\propto$ the principal.

326. One quantity varies *directly* as another, when the two increase or decrease together by a constant ratio.

Sometimes, for the sake of brevity, we say simply one quantity varies as another.

Thus, if a man agrees to work for a certain sum per day, the amount of his wages varies as the number of days during which he works; for, as the number of days increases or decreases, the amount of his wages will increase or decrease, and in the *same ratio.*

327. One quantity varies *inversely* as another, when the first varies as the reciprocal of the second.

Thus, if a courier has a fixed route, the time in which he will pass over it varies inversely as his speed. That is, if he *double* his speed, he will go in *half* the time; and so on.

328. One quantity varies as two others jointly, when it has a constant ratio to the product of the two.

Thus, the wages to be received by a workman will *vary* as the number of days he has worked, and the wages per day, *jointly;* for if either the number of days, or the wages per day, be doubled, trebled, etc., so as to double or treble, etc., their product, the whole wages will be doubled, trebled, etc.; that is, changed by a constant ratio.

329. One quantity varies directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, in physics, the attraction of a planetary body varies directly as the quantity of matter, and inversely as the square of the distance.

330. *If* A *varies as* B, *then* A *is equal to* B *multiplied by some constant quantity.*

Let a and b denote one pair of corresponding values of two quantities, and A and B denote any other pair; then, by Art. 326, $\frac{A}{a} = \frac{B}{b}$. Hence,

$$A = \frac{a}{b} B = m b,$$

where m is equal to the constant $\frac{a}{b}$.

331. Hence, also, *when any quantity varies as another, if any two pairs of the given values be taken, the four will be proportional.*

For, if $A \propto B$, and a and b be any pair of values of A and B, then for any other values of A and B, we have

$$A : B = m = a : b.$$

332. The terms used in Variation may now be distinguished as follows: —

1. *If* $\mathrm{A} = \mathrm{m\,B}$, *then* A *varies directly as* B;
2. *If* $\mathrm{A} = \frac{\mathrm{m}}{\mathrm{B}}$, *then* A *varies inversely as* B;
3. *If* $\mathrm{A} = \mathrm{m\,B\,C}$, *then* A *varies jointly as* B *and* C;
4. *If* $\mathrm{A} = \mathrm{m}\frac{\mathrm{B}}{\mathrm{C}}$, *then* A *varies directly as* B, *and inversely as* C.

Problems in variation, in general, are readily solved by converting the variation into an equation.

EXAMPLES.

1. Given that $y \propto x$; when $x = 2$, $y = 10$; required the equation between x and y. Ans. $y = 5x$.

2. Given that $y \propto x$, and that $y = 2$ when $x = 1$; what will be the value of y when $x = 2$? Ans. 4.

3. If y varies as z, and $y = 24$ when $z = 3$, find the equation between y and z. Ans. $y = 8z$.

4. If x varies inversely as y, and $x = 4$ when $y = 2$; what is the value of x when $y = 6$? Ans. $1\frac{1}{3}$.

5. Given that z varies jointly as x and y, and that $z = 1$ when $x = 1$ and $y = 1$; find the value of z when $x = 2$ and $y = 2$. Ans. 4.

6. If $y =$ the sum of two quantities, whereof one is constant, and the other varies as xy, and when $x = 2$, $y = -2\frac{1}{3}$, but when $x = -2$, $y = 1$, express y in terms of x.

Ans. $y = \frac{14}{4 - 5x}$.

7. Two circular plates of gold, each an inch thick, the diameters of which are 6 inches and 8 inches, respectively, are melted and formed into a single circular plate one inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter. Ans. 10.

8. Given that the illumination from a source of light varies inversely as the square of the distance; how much farther from a candle must a book, which is now 3 inches off, be removed, so as to receive just half as much light?

Ans. $3(\sqrt{2} - 1)$ inches, or nearly $1\frac{1}{4}$ inches.

9. A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of cars attached. With four cars its speed is 20 miles an hour. Find the greatest number of cars which the engine can move. Ans. 143.

PROGRESSIONS.

333. A PROGRESSION is a series, or a succession of terms increasing or decreasing according to some uniform law.

There may be distinguished three kinds of Progression: Arithmetical, Geometrical, and Harmonical.

ARITHMETICAL PROGRESSION.

334. An ARITHMETICAL PROGRESSION is a series that increases or decreases from term to term by a common difference.

Its formation may be considered to be by the continual *addition* of the common difference; therefore, when the series is *increasing*, the common difference will be *positive*, and when *decreasing*, it will be *negative*. Thus,

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \ldots\ldots$$

is an *increasing* arithmetical progression, in which the common difference is $+2$; and

$$19, \quad 17, \quad 15, \quad 13, \quad 11, \quad 9, \ldots\ldots$$

is a *decreasing* arithmetical progression, in which the common difference is -2.

335. *In an Arithmetical Progression, the* LAST TERM *is equal to the first term plus the product of the common difference by the number of terms less* 1.

Let a denote the first term, d the common difference, and l the last term; then the progression will be

$$a, \quad (a+d), \quad (a+2d), \quad (a+3d), \ldots\ldots$$

That is, the nth term of the series will be

$$a + (n-1)\,d.$$

Also, by putting l for the nth term, we have

$$l = a + (n - 1)\, d,$$

where d is either positive or negative, according as the series is an increasing or decreasing one.

336. *In an Arithmetical Progression, the* SUM OF THE TERMS *is equal to the product of one half the number of terms by the sum of the extremes.*

Let S denote the sum of the terms; then

$$S = a + (a + d) + (a + 2d) + \ldots\ldots + l,$$

or, writing the terms in the reverse order,

$$S = l + (l - d) + (l - 2d) + \ldots\ldots + a.$$

Therefore, by adding these equations, term by term,

$$2S = (a + l) + (a + l) + (a + l) + \ldots\ldots + (a + l).$$

Here, $(a + l)$ is taken as many times as there are terms, or n times; whence,

$$2S = n\,(a + l), \quad \text{or} \quad S = \tfrac{1}{2} n\,(a + l).$$

In the course of the demonstration, it may, also, be readily seen, that

The sum of any two terms equidistant from the extremes, is equal to the sum of the extremes.

337. *To insert a given number of arithmetical means between two given terms.*

Let a and l be the two given terms, and m the number of terms to be inserted. Then we are to find $m + 2$ terms in arithmetical progression, a being the first term, and l the last. Let, now, d denote the common difference; then $l = a + (m + 1)\, d$; whence $d = \dfrac{l - a}{m + 1}$. This determines d, and the m required means will be

$$a + d,\ a + 2d,\ a + 3d,\ \ldots\ldots\ a + md.$$

That is,

The given first term plus the common difference will give the first of the required means; the first mean plus the common difference will give the second mean; and so on.

338. The formulas established in Arts. 335, 336,

$$l = a + (n-1)\,d, \qquad (1)$$

$$S = \tfrac{1}{2}n\,(a+l), \qquad (2)$$

are *fundamental;* and, since they constitute two independent equations, together containing all the five elements of an arithmetical progression, when any three of these are given, the other two may be readily determined. Thus, from these two equations we deduce other formulas; and there are in all twenty, as exhibited in the following

TABLE.

No.	Given.	Required.	Formulas.
1	a, d, n	l, S	$l = a + (n-1)\,d$; $\quad S = \frac{1}{2}n\,[2a + (n-1)\,d]$.
2	l, d, n	a, S	$a = l - (n-1)\,d$; $\quad S = \frac{1}{2}n\,[2l - (n-1)\,d]$.
3	d, n, S	a, l	$a = \dfrac{2S - n(n-1)\,d}{2n}$; $\quad l = \dfrac{2S + n(n-1)\,d}{2n}$.
4	a, n, l	d, S	$d = \dfrac{l-a}{n-1}$; $\quad S = \frac{1}{2}n\,(a+l)$.
5	a, n, S	d, l	$d = \dfrac{2(S - an)}{n(n-1)}$; $\quad l = \dfrac{2S}{n} - a$.
6	l, n, S	d, a	$d = \dfrac{2(nl - S)}{n(n-1)}$; $\quad a = \dfrac{2S}{n} - l$.
7	a, d, l	n, S	$n = \dfrac{l-a}{d} + 1$; $\quad S = \dfrac{(l+a)(l-a+d)}{2d}$.
8	a, l, S	n, d	$n = \dfrac{2S}{a+l}$; $\quad d = \dfrac{(l+a)(l-a)}{2S - (l+a)}$.
9	a, d, S	n, l	$n = \dfrac{d - 2a \pm \sqrt{(d-2a)^2 + 8dS}}{2d}$; $\quad l = a + (n-1)\,d$.
10	l, d, S	n, a	$n = \dfrac{d + 2l \pm \sqrt{(d+2l)^2 - 8dS}}{2d}$; $\quad a = l - (n-1)\,d$.

Hence, for the solution of questions in Arithmetical Progression, the following

RULE.

Substitute in the fundamental formulas, or such as may be deduced from them, the given quantities and reduce the result.

EXAMPLES.

1. If the first term of an arithmetical progression is 5, and the common difference 3, what is the 7th term?

Here, $a = 5$, $d = 3$, and $n = 7$, to find l.

Whence $l = 5 + (7 - 1)\,3 = 23$, Ans.

2. Required the 4th term of an arithmetical progression, of which the first term is 10, and the common difference -2. Ans. 4.

3. If the first term is $\frac{1}{3}$, the common difference $-\frac{1}{12}$, and the sum of the series $-1\frac{1}{2}$, what is the number of terms? Ans. 12.

4. The first term is 3, the number of terms 20, and the sum of the terms 440. Required the common difference. Ans. 2.

5. The sum of n terms of a series, of which the common difference is 3, is $\frac{3n^2}{2} + 2n$. What is the first term? Ans. $\frac{7}{2}$.

6. Insert three arithmetical means between -9 and 18. Ans. $-2\frac{1}{4}$, $4\frac{1}{2}$, $11\frac{1}{4}$.

7. Required the first and last terms of a series of which the common difference is 5, the number of terms 6, and the sum 321. Ans. 41 and 66.

8. Find the sum of the odd numbers from 1 to 100. Ans. 2500.

9. A debt can be discharged in a year by paying \$1 the first week, \$3 the second, \$5 the third, and so on; required the last payment, and the amount of the debt.

Ans. Last payment, \$103; amount, \$2704.

10. A person saves \$270 the first year, \$210 the second, and so on. In how many years will a person who saves every year \$180 have saved as much as he?

Ans. 4.

11. Two persons start together. The one travels ten leagues a day, the other eight leagues the first day, which he augments daily by half a league. After how many days, and at what distance from the point of departure, will they come together?

Ans. After 9 days, at a distance of 90 leagues.

GEOMETRICAL PROGRESSION.

339. A GEOMETRICAL PROGRESSION is a series, each term of which is equal to the preceding one, multiplied by a constant factor.

The constant factor is called the *ratio* of the progression.

Since the successive terms of the progression may be considered as derived from the first by continually *multiplying* it by the ratio, therefore, if the first term is positive, the series is *increasing* when the ratio is *greater* than 1, but the series is *decreasing* when the ratio is *less* than 1. Thus,

$$3,\ 6,\ 12,\ 24,\ 48, \ldots\ldots$$

is an *increasing* geometrical progression, in which the first term is 3, and the ratio 2; and

$$9,\ 3,\ 1,\ \tfrac{1}{3},\ \tfrac{1}{9}, \ldots\ldots$$

is a *decreasing* geometrical progression, in which the ratio is $\frac{1}{3}$.

340. *In a Geometrical Progression, the* LAST TERM *is equal to the product of the first term by the ratio raised to a power whose exponent is one less than the number of terms.*

Let a denote the first term, r the ratio, and l the last term; then the progression will be

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots$$

That is, the nth term of the series will be

$$ar^{n-1}.$$

Also, by putting l for the nth term, we have

$$l = ar^{n-1}.$$

341. *To find the* SUM *of a given number of terms in a Geometrical Progression.*

Let S denote the sum of the terms; then

$$S = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying by r,

$$rS = ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-1} + ar^n. \quad (2)$$

Subtracting (1) from (2), and factoring,

$$(r-1)\,S = a\,(r^n - 1) \quad (3)$$

Therefore,

$$S = \frac{a\,(r^n - 1)}{r - 1} \quad (4)$$

Again, if l denote the last term, by Art. 340,

$$l = ar^{n-1}$$

Multiplying by r,

$$rl = ar^n$$

Substituting the value of ar^n in (4),

$$S = \frac{rl - a}{r - 1}$$

That is,

The sum is equal to the difference between the first term and the product of the last term by the ratio, divided by the difference between 1 *and the ratio.*

342. The limit to which the sum of a decreasing geometrical series approaches, as the number of terms becomes larger and larger, is called the *sum of the series to infinity.*

We may write the value of S, in equation (4), Art. 341, by changing the signs of the terms of the fraction, under the equivalent form,

$$S = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now, suppose r less than 1; then when the number of terms, n, becomes infinitely great, or equal to ∞, the fraction $\frac{ar^n}{1-r}$ must become infinitely small, or equal to 0, and may be rejected. Hence, when the number of terms in a decreasing geometrical series is infinite,

$$S = \frac{a}{1-r}.$$

That is,

The sum of the terms of a decreasing geometrical series to infinity is equal to the first term divided by 1 *less the ratio.*

343. *To insert a given number of geometrical means between two given terms.*

Let a and l be the two given terms, and m the number of terms to be inserted. Then we are to find $m+2$ terms in geometrical progression, a being the first term, and l the last. Let, now, r denote the ratio; then

$$l = ar^{m+1}; \text{ whence } r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}}.$$

This determines r, and the m required means will be

$$ar,\ ar^2,\ ar^3,\ \ldots\ ar^m; \text{ that is,}$$

The given first term multiplied by the ratio will give the first of the required means; the first mean multiplied by the ratio will give the second mean; and so on.

344. The formulas established in Arts. 340, 341,

$$l = a r^{n-1}, \qquad (1)$$

$$S = \frac{rl - a}{r - 1}, \qquad (2)$$

are *fundamental;* and since they contain the five elements, if any three of these are given, formulas may be deduced for finding the other two, as in the following

TABLE.

No.	Given.	Required.	Formulas.	
1	a, r, n	l, S	$l = a r^{n-1};$	$S = \frac{a r^n - a}{r - 1}.$
2	l, r, n	a, S	$a = \frac{l}{r^{n-1}};$	$S = \frac{l r^n - l}{r^n - r^{n-1}}.$
3	r, n, S	a, l	$a = \frac{(r-1)S}{r^n - 1};$	$l = \frac{(r-1) S r^{n-1}}{r^n - 1}.$
4	a, n, l	r, S	$r = \sqrt[n-1]{\frac{l}{a}};$	$S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
5	a, n, S	r, l	$a r^n - r S = a - S;$	$l (S-l)^{n-1} = a (S-a)^{n-1}.$
6	l, n, S	r, a	$(S-l) r^n - S r^{n-1} = -l;$	$a (S-a)^{n-1} = l (S-l)^{n-1}.$
7	a, r, l	n, S	$n = \frac{\log l - \log a}{\log r} + 1;$	$S = \frac{l r - a}{r - 1}.$
8	a, l, S	n, r	$n = \frac{\log l - \log a}{\log (S-a) - \log (S-l)} + 1;$	$r = \frac{S-a}{S-l}.$
9	a, r, S	n, l	$n = \frac{\log [a + (r-1) S] - \log a}{\log r};$	$l = \frac{a + (r-1) S}{r}.$
10	l, r, S	n, a	$n = \frac{\log l - \log [lr - (r-1) S]}{\log r} + 1;$	$a = lr - (r-1) S.$

The formulas for n, exhibited in the table for convenience of reference, require for their solution methods not yet given, but which will be presented in their proper place; and those containing the nth degree of the unknown quantity may present difficulty, unless n is less than 3.

Hence, for the solution of questions in Geometrical Progression, we have the following

RULE.

Substitute in the appropriate formula the given quantities, and reduce the result.

EXAMPLES.

1. If the first term of a geometrical progression is 7, the ratio 3, and the number of terms 5, what is the last term?

Here, $a = 7$, $r = 3$, and $n = 5$, to find l.

Whence $l = 7\ (3)^{5-1} = 7 \times 3^4 = 567$, Ans.

2. If the first term of a geometrical progression is 7, the ratio 3, and the number of terms 5, what is the sum of the series? Ans. 847.

3. If the ratio is 2, the number of terms 6, and the last term 128, what is the first term? Ans. 4.

4. If the first term is 2, the last term 4374, and the number of terms 8, what is the ratio? Ans. 3.

5. Insert three geometrical means between $\frac{1}{2}$ and 128.

Ans. 2, 8, 32.

6. If the first term is 2, the ratio 4, and the number of terms 12, what are the last term, and the sum of the series?

Ans. Last term, 8388608; sum of the series, 11184810.

7. Required the sum of the series $1, \frac{1}{3}, \frac{1}{9}, \ldots\ldots$ to infinity. Ans. $1\frac{1}{2}$.

8. Find the sum of $\frac{3}{2}, 1, \frac{2}{3}, \ldots\ldots$ to six terms.

Ans. $4\frac{17}{162}$.

9. Find the value of the decimal .333 . . . . . to infinity.

Here, $.333 \ldots\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} \ldots\ldots$

Whence, $S = \frac{3}{10} \div (1 - \frac{1}{10}) = \frac{3}{9} = \frac{1}{3}$, Ans.

10. Find the value of .1212 . . . . . to infinity.

Ans. $\frac{4}{33}$.

11. Required the sum of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\ldots$ to infinity. Ans. $\frac{2}{3}$.

12. A person who saved every year half as much again as he saved the previous year, had in seven years saved $2059. How much did he save the first year?

Ans. $64.

13. A gentleman, by agreement, boarded 9 days, paying 3 cents for the first day, 9 cents for the second day, 27 cents for the third day, and so on, in this ratio. Required the cost. Ans. $295.23.

HARMONICAL PROGRESSION.

345. Three or more quantities are said to be in HARMONICAL PROGRESSION, when their reciprocals form an arithmetical progression.

Thus, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$

are in harmonical progression, because their reciprocals,

$1, 3, 5, 7, \ldots\ldots$

are in arithmetical progression.

346. *If three consecutive terms of an Harmonical Progression be taken, the first has the same ratio to the third that the difference of the first and second has to the difference of the second and third.*

For, if a, b, c are in harmonical progression, their reciprocals, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, will be in arithmetical progression; therefore,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$$

Clearing of fractions, we have

$$ab - ac = ac - bc$$

Or, $$a(b - c) = c(a - b)$$

Dividing by $c(b - c)$, $$\frac{a}{c} = \frac{a - b}{b - c}$$

Whence, $$a : c :: a - b : b - c.$$

347. From the preceding it follows that every series of quantities in harmonical progression may be easily converted into an arithmetical progression, and then the rules of the latter may be applied. There will be found, however, no general expression for the sum of an harmonical series. Thus,

1. Given the first two terms of an harmonical progression, to find the nth term.

Let a and b be the first two terms, and l the nth term; then $\frac{1}{a}$ and $\frac{1}{b}$ are the first two terms of an arithmetical progression, and $\frac{1}{l}$ is its nth term; therefore,

$$d = \frac{1}{b} - \frac{1}{a} = \frac{a - b}{ab}$$

Whence,

$$\frac{1}{l} = \frac{1}{a} + (n - 1)\frac{a - b}{ab} = \frac{(n - 1)a - (n - 2)b}{ab}$$

Therefore, $$l = \frac{ab}{(n - 1)a - (n - 2)b}. \qquad (1)$$

2. To insert m harmonical means between a and l.

Here, if d be the common difference of the reciprocals of the terms, we have

$$\frac{1}{l} = \frac{1}{a} + (n-1)\,d;$$

or,
$$d = \frac{a-l}{(n-1)\,al} = \frac{a-l}{(m+1)\,al}; \qquad (2)$$

whence the arithmetical progression is found; and by inverting its terms, the harmonical means will be ascertained.

EXAMPLES.

1. The first term of an harmonical progression is 1, and the second term $\frac{1}{2}$; find the fifth term. Ans. $\frac{1}{5}$.

2. Insert two harmonical means between $\frac{1}{6}$ and $\frac{1}{12}$; and also between 4 and 5. Ans. $\frac{1}{8}$, $\frac{1}{10}$; $4\frac{2}{7}$, $4\frac{8}{13}$.

3. Continue the series 3, $\frac{6}{5}$, $\frac{3}{4}$, for two terms.
Ans. $\frac{6}{11}$, $\frac{6}{14}$.

PROBLEMS

REQUIRING THE APPLICATION OF THE PRINCIPLES OF THE PROGRESSIONS.

1. It is required to find four numbers in arithmetical progression, such that the product of the extremes shall be 45, and the product of the means 77.

SOLUTION.

Let x be the first term, and y the common difference; then the numbers will be

$$x,\ x+y,\ x+2y,\ x+3y;$$

and by the conditions,

$$x^2 + 3xy = 45 \quad (1)$$
$$x^2 + 3xy + 2y^2 = 77 \quad (2)$$

Subtracting (1) from (2), $2y^2 = 32$

Whence, $y = \pm 4$

Substituting the value of y in (1), we have

$$x^2 \pm 12x = 45$$

Whence, $x = \pm 3$, or ± 15.

Hence, the four numbers are 3, 7, 11, and 15.

2. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

SOLUTION.

Let x and y denote the two extremes; then $\sqrt{xy}$ is the mean, and by the conditions,

$$x + \sqrt{xy} + y = 14 \quad (1)$$
$$x^2 + xy + y^2 = 84 \quad (2)$$

Dividing (2) by (1), $x - \sqrt{xy} + y = 6$ (3)

Adding (1) and (3), $x + y = 10$ (4)

Subtracting (3) from (4), $\sqrt{xy} = 4$ (5)

Or, $xy = 16$ (6)

Therefore, $(x + y)^2 - 4xy = 100 - 64 = 36$ (7)

Whence, $x - y = \pm 6$ (8)

From (4) and (8), $x = 8$, or 2; $y = 2$, or 8.

Hence, the three numbers are 2, 4, and 8.

3. Find four numbers in arithmetical progression, such that the sum of the squares of the first two terms shall be 34, and the sum of the squares of the last two 130.

Ans. 3, 5, 7, and 9.

4. The sum of three numbers in harmonical proportion is 13, and the product of their extremes is 18. What are the numbers?

Ans. 3, 4, and 6.

Let x and y denote the extremes; then $\frac{2xy}{x+y}$ is the mean.

5. A gentleman set out from Boston for New York. He traveled 25 miles the first day, 20 miles the second day, each day traveling 5 miles less than the preceding. How far was he from Boston at the end of the eleventh day?

Ans.

6. Suppose a number of stones were laid a rod distant from each other for twenty miles, and the first stone a rod from a basket. What length of ground will that man travel over who gathers them up singly, returning with them, one by one, to the basket?

Ans. 128,060 miles 2 rods.

7. If a number of workmen be 2 days in raising the tenth foot of a tower which is to be 100 feet high, how long will they be in building the tower, the time of raising any foot being in proportion to its height?

Ans. 1010 days.

8. Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre, and they all together can finish the job in 60 days. In how many days can each, separately, reap an acre? Ans. 2, 3, 4, 5, 6, days.

9. Suppose the elastic power of a ball that falls from a height of 100 feet, to be such as to cause it to rise 0.9375 of the height from which it fell, and to continue in this way diminishing the height to which it will rise, in geometrical progression, till it comes to rest. How far will it have moved? Ans. 3100 feet.

10. If a man travel 20 miles the first day, 15 miles the second, and so continue to travel 5 miles less each day, how far will he have advanced on his journey at the end of the 8th day? Ans. 20 miles.

11. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum

of the squares of the means is 136. What are the numbers? Ans. 2, 6, 10, and 14.

12. After A had traveled for $2\frac{3}{4}$ hours, at the rate of 4 miles an hour, B set out to overtake him, and went $4\frac{1}{2}$ miles the first hour, $4\frac{3}{4}$ the second, 5 the third, and so on, increasing his speed a quarter of a mile every hour. In how many hours would he overtake A?

Ans. 8 hours.

13. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

Ans. 5, 10, 20, and 40.

14. The greatest of three numbers in harmonical progression is the product of the other two; but, if one be added to each, the greatest becomes the sum of the other two. What are the numbers? Ans. 2, 3, and 6.

PERMUTATIONS AND COMBINATIONS.

348. The different orders in which quantities can be arranged are called their PERMUTATIONS.

Thus, the *permutations* of the letters a, b, c, taken *two* at a time, are

$$ab,\ ba;\ ac,\ ca;\ bc,\ cb;$$

and taken *three* at a time, are

$$abc,\ acb;\ bac,\ bca;\ cab,\ cba.$$

349. The COMBINATIONS of quantities are the different collections that can be formed out of them, without regarding the order in which they are placed.

Thus, the *combinations* of the letters a, b, c, taken two at a time, are

$$ab,\ ac,\ bc,$$

ab and ba, though different permutations, forming the same combination.

350. *To find the number of permutations of* n *quantities taken* r *together.*

Let $a, b, c, d, \ldots\ldots k$ be n quantities. The number of permutations of these n quantities, taken singly, is evidently n.

The number of permutations of n quantities, taken *two* together, is $n\ (n-1)$.

For, since there are n quantities, if we put one of these, as a, before the others, we obtain

$$ab,\ ac,\ ad,\ \ldots\ldots\ ak,$$

or $n-1$ permutations in which a stands first. Proceeding in the same manner with b, we have $n-1$ permutations in which b stands first. Similarly, there will be $n-1$ permutations in which c stands first, and so on. Hence, taking *two* together, the whole number of permutations will be

$$n\ (n-1).$$

The number of permutations of n quantities, taken *three* together, is $n\ (n-1)\ (n-2)$.

For, it has just been shown that out of n quantities we can form $n\ (n-1)$ permutations, each of two quantities. Hence, out of $n-1$ quantities we can form $(n-1)\ (n-2)$ permutations, each of two quantities; put a before each of these and we have $(n-1)\ (n-2)$ permutations, each of three quantities, in which a stands first. Similarly, there are $(n-1)\ (n-2)$ permutations, each of three quantities, in which b stands first; as many in which c

stands first; and so on. Hence, taking *three* together, the whole number of permutations will be

$$n\ (n-1)\ (n-2).$$

In like manner, we can prove that the number of permutations of n quantities, taken *four* together, will be

$$n\ (n-1)\ (n-2)\ (n-3).$$

In examining the results obtained, we perceive such a relation existing between each class of permutations and the numerical part of its corresponding expression, that the negative number in the last factor is 1 less than the number of quantities taken together. Hence, we may conclude that the number of permutations of n quantities, taken r together, is

$$n\ (n-1)\ (n-2)\ \ldots\ldots\ (n-r+1). \qquad (1)$$

351. If we suppose all the quantities to be taken together, or $r = n$, formula (1) becomes

$$n\ (n-1)\ (n-2)\ (n-3)\ \ldots\ldots\ 1, \qquad (2)$$

or, by inverting the order of the factors,

$$1 \times 2 \times 3\ \ldots\ldots\ (n-1)\ n. \qquad (3)$$

That is,

The number of permutations of n *letters, taken* n *together, is equal to the product of the natural numbers from* 1 *up to* n.

For the sake of brevity, this result is often denoted by $\underline{|n}$; thus $\underline{|n}$ denotes the product of the natural numbers from 1 to n inclusive.

352. *To find the number of combinations of* n *quantities, taken* r *together.*

The number of combinations of n quantities, taken singly, is evidently n.

The number of combinations of n quantities, taken *two* together, is $\frac{n\,(n-1)}{1 \times 2}$.

For the number of permutations of n quantities, taken two together, is $n\,(n-1)$; but each combination of two quantities admits of two permutations; hence, the number of combinations is the number of permutations divided by 2.

The number of combinations of n quantities, taken *three* together, is $\frac{n\,(n-1)\,(n-2)}{1 \times 2 \times 3}$.

For, there are $n\,(n-1)\,(n-2)$ permutations in n things taken three together; but each combination of three quantities admits of $1 \times 2 \times 3$ permutations; hence, the number of combinations is that of the permutations divided by $1 \times 2 \times 3$.

In like manner, it may be shown that the number of combinations of n things taken r together, is

$$\frac{n\,(n-1)\,(n-2)\,\ldots\,(n-r+1)}{1 \times 2 \times 3 \times \ldots \times r}. \qquad (4)$$

353. *The number of combinations of* n *quantities taken* r *together, is the same as the number of combinations of* n *quantities taken* n — r *together.*

For, it is evident that for every combination of r quantities which we take out of n quantities, we leave one combination of $n-r$ quantities, which contains the remaining quantities.

EXAMPLES.

1. How many changes can be rung with 10 bells, taken 7 at a time?

Here, $n = 10$, $r = 7$, and $n - r + 1 = 4$;

then, by formula (1),

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 604800, \text{ Ans.}$$

2. In how many different orders may 7 persons be seated at table?

Here, by formula (3),

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040, \text{ Ans.}$$

3. How many different combinations can be made with 5 letters out of 8?

Here, $n = 8$, $r = 5$, and $n - r + 1 = 4$;

then, by formula (4),

$$\frac{8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5} = 56, \text{ Ans.}$$

4. How often can 4 students change their places in a class of 8, so as not to preserve the same order?

Ans. 1680.

5. From a company of 40 soldiers, how many different days can a different picket of 6 men be taken?

Ans. 3838380.

BINOMIAL THEOREM.

354. The Binomial Theorem is a formula, by means of which any binomial may be raised to any given power, without the use of the ordinary process of involution.

We have already seen that $(x + a)^2 = x^2 + 2xa + a^2$, and that $(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$; our object now is to find an expression equal to $(x + a)^n$, where n is any *positive integer*.

355. By actual multiplication we have

$$(x + a_1)(x + a_2) = x^2 + (a_1 + a_2)\,x + a_1 a_2,$$

$$(x + a_1)(x + a_2)(x + a_3) = x^3 + (a_1 + a_2 + a_3)\,x^2 + (a_1 a_2 + a_2 a_3 + a_1 a_3)\,x + a_1 a_2 a_3,$$

$$(x + a_1)(x + a_2)(x + a_3)(x + a_4) = x^4 + (a_1 + a_2 + a_3 + a_4)\,x^3 + (a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4)\,x^2 + (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4)\,x + a_1 a_2 a_3 a_4.$$

Now, in these results we see that the following laws hold : —

1. *The number of terms on the right is one more than the number of binomial factors which are multiplied together.*

2. *The exponent of* x *in the first term is the same as the number of binomial factors, and in the succeeding terms each exponent is less than that of the preceding term by unity.*

3. *The coefficient of the first term is unity; the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third term is the sum of the products of the second terms of the binomial factors, taken two together; the coefficient of the fourth term is the sum of the products of the second terms of the binomial factors, taken three together, and so on; the last term is the product of all the second terms of the binomial factors.*

It will now be proved that these laws always hold, whatever be the number of binomial factors. Suppose the laws to hold when $n - 1$ factors are multiplied together ; that is, suppose

$$(x + a_1)(x + a_2) \ldots (x + a_{n-1}) = x^{n-1} + p_1 x^{n-2} + p_2 x^{n-3} + p_3 x^{n-4} + \ldots + p_{n-1},$$

where $p_1 =$ the sum of the terms $a_1, a_2, \ldots\ldots a_{n-1}$.

$p_2 =$ the sum of the products of these terms, taken two together.

$p_3 =$ the sum of the products of these terms, taken three together.

. . . . . .

p_{n-1} = the product of all these terms.

Multiply both members of this identical equation by another factor, $x + a_n$. Thus,

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_n) = x^n + (p_1+a_n)x^{n-1} + (p_2+p_1a_n)x^{n-2} + (p_3+p_2a_n)x^{n-3} + \ldots\ldots + p_{n-1}a_n.$$

Now,

$p_1 + a_n = a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n$
= the sum of all the terms $a_1, a_2, \ldots\ldots a_n$;

$p_2 + p_1a_n = p_2 + a_n(a_1 + a_2 + \ldots\ldots + a_{n-1})$
= the sum of the products, taken two together, of all the terms $a_1, a_2, \ldots\ldots a_n$;

$p_3 + p_2a_n = p_3 + a_n(a_1a_2 + a_2a_3 + a_1a_3 + \ldots\ldots)$
= the sum of the products, taken three together, of all the terms $a_1, a_2, \ldots\ldots a_n$.

. . . . . .

$p_{n-1}a_n$ = the product of all the terms.

Hence, if the laws hold when $n - 1$ factors are multiplied together, they hold when n factors are multiplied together; but they have been proved to hold when 4 factors occur, therefore they hold when 5 factors occur, and so on; thus they hold universally.

The result for the multiplication of n factors may be written thus,

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_n) = x^n + q_1x^{n-1} + q_2x^{n-2} + \ldots\ldots + q_n.$$

The number of terms in q_1 is obviously n; the number of terms in q_2 is the same as the number of combinations of the n quantities $a_1, a_2, \ldots\ldots a_n$, taken two together, that is, $\frac{n(n-1)}{1 \cdot 2}$; and so on. Now suppose $a_1, a_2, \ldots\ldots a_n$ each equal to a; thus q_1 becomes na, q_2 becomes $\frac{n(n-1)}{1 \cdot 2}a^2$, and so on; and we obtain (Art. 353),

$$(x+a)^n = x^n + n\,a\,x^{n-1} + \frac{n\,(n-1)}{1\,.\,2}\,a^2 x^{n-2} + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,a^3 x^{n-3}$$
$$+ \ldots\ldots\ldots + \frac{n\,(n-1)}{1\,.\,2}\,a^{n-2} x^2 + n\,a^{n-1} x + a^n;$$

or, exchanging the letters a and x,

$$(a+x)^n = a^n + n\,a^{n-1} x + \frac{n\,(n-1)}{1\,.\,2}\,a^{n-2} x^2 + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,a^{n-3} x^3$$
$$+ \ldots\ldots\ldots + \frac{n\,(n-1)}{1\,.\,2}\,a^2 x^{n-2} + n\,a\,x^{n-1} + x^n, \quad \text{(A)}$$

in which n, $\frac{n\,(n-1)}{1\,.\,2}$, etc., are considered the coefficients.

This is the *Binomial Formula.* The series on the right of the sign of equality is called the *expansion* of $(a+x)^n$.

356. In the preceding formula, suppose $a = 1$, and we have

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{1\,.\,2}\,x^2 + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,x^3 + \ldots\ldots$$
$$\ldots\ldots + \frac{n\,(n-1)}{1\,.\,2}\,x^{n-2} + n\,x^{n-1} + x^n. \quad \text{(B)}$$

357. In formula (B), suppose x changed into $-x$, and it becomes

$$(1-x)^n = 1 - nx + \frac{n\,(n-1)}{1\,.\,2}\,x^2 - \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,x^3 + \ldots\ldots \quad \text{(C)}$$

where, since every odd power of $-x$ is negative (Art. 202), the signs are alternately plus and minus.

358. The expansion of a binomial can always be reduced to the case in which one of the two terms of the binomial is 1. For,

$$(a+x)^n = a^n \left(1 + \frac{x}{a}\right)^n = a^n\,(1+y)^n, \quad \text{if } y = \frac{x}{a}.$$

Expand now $(1+y)^n$, and multiply each term by a^n, and we have the expansion of $(a+x)^n$.

359. If in formula (B), which is true for all values of x, we put $x = 1$, we have

$$2^n = 1 + n + \frac{n(n-1)}{1 \cdot 2} + \ldots\ldots\ldots + \frac{n(n-1)}{1 \cdot 2} + n + 1;$$

that is, when both terms of a binomial are positive, the sum of the coefficients of the expansion is equal to 2 raised to the same power as the binomial.

In like manner it may be shown, by putting $x = 1$ in formula (C), that, when one of the terms of the binomial is negative, the sum of the coefficients of the expansion is equal to zero.

360. From the preceding principles may be readily deduced some important precepts, with relation to the expansion of any binomial, $(a + x)^n$.

1. *The leading letter enters all the terms except the last, and the following letter all except the first.*

2. *In the successive terms, the exponents of the leading letter, beginning with the exponent of the power, decrease by* 1, *while those of the following letter, beginning in the second term with* 1, *increase by* 1; *and the sum of the exponents in any term is equal to the exponent of the given power.* Thus, the exponents are

$$a^n,\ a^{n-1}x,\ a^{n-2}x^2,\ a^{n-3}x^3,\ \ldots\ldots\ a^2x^{n-2},\ ax^{n-1},\ x^n,$$

where the sum in any term is n, or the exponent of the power.

3. *If the coefficient of any term be multiplied by the exponent of the leading letter of the same term, and the product divided by the number of the term from the left, the result will be the coefficient of the term next following.* Thus,

The coefficient of the first term is 1, of the second $\frac{1 \times n}{1}$, or n, of the third $\frac{n(n-1)}{2}$, and so on.

4. *The number of terms is always one more than the exponent of the power.*

5. *If both terms of the binomial are positive, all the terms of the power are positive; but if the second term of the binomial is negative, the even terms of the power will be negative.*

6. *The coefficients of any two terms taken equidistant from the beginning and end of the expansion are the same.*

For, the rth and $(n - r + 1)$th terms are equidistant from the first and last terms, and their coefficients are, respectively, the number of combinations of n quantities taken $r - 1$, and $n - (r - 1)$, together (Art. 353).

7. *The general term of the expansion, or that from which the others can be deduced,* is

$$r\text{th term} = \frac{n(n-1)(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \ldots (r-1)} a^{n-r+1} x^{r-1}.$$

For, the expansion of $(a + x)^n$ exhibits the exponent of a in any term, n diminished by a number which is 1 less than the number of the term; that is, the exponent of a for the second term is $n - 1$, for the third $n - 2$, and so on; for the rth term it would be $n - r + 1$. Also, the last factor of the coefficient of any term has for its numerator n diminished by a number which is two less than the number of the term, and for its denominator a number 1 less than the number of the term. Finally, the exponent of x is 1 less than the number of the term.

361. By the sixth precept, the latter half of an expansion may be written out from the first half; and by means of the seventh precept, any single term of the expansion may be found, independently of the rest.

EXAMPLES.

1. Raise $a + x$ to the fifth power.

By 2, Art. 360, the terms without the coefficients are

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5;$$

and by 3, Art. 360, the coefficients are

$$1,\ \frac{1\times 5}{1},\ \frac{5\times 4}{2},\ \frac{10\times 3}{3},\ \frac{10\times 2}{4},\ \frac{5\times 1}{5};$$

or, 1, 5, 10, 10, 5, 1.

whence, by prefixing the coefficients, we obtain

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5, \text{ Ans.}$$

Here, $n+1$, or the number of terms, is equal to 6; therefore, it was only necessary to determine the coefficients for the first 3 terms, and repeat the same numbers in the reverse order for the others.

2. Raise $a - b$ to the fifth power.

Ans. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

3. Find the 10th term of the expansion $(a - x)^{18}$.

Here, $n = 18$, and $r = 10$; and substituting these in the general term (7, Art. 360), we have

$$-\frac{18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}a^9x^9 = -48620\,a^9x^9.$$

4. Find the 49th term of $(a - x)^{50}$.

Ans. $1225\,a^2x^{48}$.

5. Raise $a + x$ to the seventh power.

Ans. $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$.

6. Find the 9th term in the development of $(x + a)^{10}$.

Ans. $45\,x^2a^8$.

7. Raise $1 + x$ to the sixth power.

Ans. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

362. If either or both terms of a binomial have coefficients or exponents, the binomial formula may still be applied.

8. Raise $c^2 + yz$ to the fifth power.

Here, by inclosing c^2 and yz, each in a parenthesis, they may be treated as a single literal quantity. Thus,

$$(c^2 + yz)^5 = (c^2)^5 + 5\,(c^2)^4\,yz + 10\,(c^2)^3\,(yz)^2 + 10\,(c^2)^2\,(yz)^3 + 5\,c^2\,(yz)^4 + (yz)^5 =$$
$$c^{10} + 5\,c^8\,yz + 10\,c^6\,y^2z^2 + 10\,c^4\,y^3z^3 + 5\,c^2\,y^4z^4 + y^5z^5, \text{ Ans.}$$

9. Raise $1 + x^2$ to the seventh power.

Ans. $1 + 7x^2 + 21x^4 + 35x^6 + 35x^8 + 21x^{10} + 7x^{12} + x^{14}$.

10. Required the middle term of $(m^2 + n^3)^8$.

Ans. $70\,m^8\,n^{12}$.

11. Raise $3a^2 - 2b^3$ to the sixth power.

Ans. $729\,a^{12} - 2916\,a^{10}\,b^3 + 4860\,a^8\,b^6 - 4320\,a^6\,b^9 + 2160\,a^4\,b^{12} - 576\,a^2\,b^{15} + 64\,b^{18}$.

12. Required the expansion of $\left(1 - \frac{1}{x}\right)^7$.

Ans. $1 - \frac{7}{x} + \frac{21}{x^2} - \frac{35}{x^3} + \frac{35}{x^4} - \frac{21}{x^5} + \frac{7}{x^6} - \frac{1}{x^7}$.

13. Find the term which involves $a^{13}b^7$ in the expansion of $(a^3 + 3ab)^9$. Ans. $78732\,a^{13}\,b^7$.

14. Write down the 2001st term of $\left(a^{\frac{3}{10}} + x^{\frac{3}{10}}\right)^{2002}$.

Ans. $\frac{2002 \times 2001}{1 \times 2}\,a^{\frac{3}{5}}\,x^{600}$.

15. Raise $1 - \frac{1}{2}x$ to the tenth power.

Ans. $1 - 5x + \frac{45}{4}x^2 - 15x^3 + \frac{105}{8}x^4 - \frac{63}{8}x^5 + \frac{105}{32}x^8 - \frac{15}{16}x^7 + \frac{45}{256}x^8 - \frac{5}{256}x^9 + \frac{1}{1024}x^{10}$.

363. The binomial formula may be applied to the expansion of any polynomial whatever, since, by means of the parenthesis, it may be treated as a binomial.

16. Raise $2a - b + c^2$ to the third power.

Put $2a - b = a$, and $c^2 = x$; then

$$(2a - b + c^2)^3 = (a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

Substituting for a its equal, $2a - b$, and for x its equal, c^2, we have

$$(2a - b + c^2)^3 = (2a - b)^3 + 3(2a - b)^2c^2 + 3(2a - b)c^4 + c^6.$$

Performing the operations indicated in the second member,

$$(2a - b + c^2)^3 = 8a^3 - 12a^2b + 6ab^2 - b^3 + 12a^2c^2 - 12abc^2 + 3b^2c^2 + 6ac^4 - 3bc^4 + c^6, \text{ Ans.}$$

17. Required the term involving $a^2b^2c^2$ in the expansion of $(a + b + c)^6$. Ans. $90a^2b^2c^2$.

18. Expand $(a + 2b - c)^3$ by the binomial formula.

Ans. $a^3 + 6a^2b + 12ab^2 + 8b^3 - 3a^2c - 12abc - 12b^2c + 3ac^2 + 6bc^2 - c^3$.

SERIES.

364. A SERIES is a succession of terms, so related that each may be derived from one or more of the others, in accordance with some fixed law.

The simpler forms of series have already been exhibited in the progressions.

365. A FINITE SERIES is one having a finite number of terms.

366. An INFINITE SERIES is one having an infinite number of terms, as

$$1 + x + x^2 + x^3 + \ldots\ldots$$

367. An infinite series is said to be *convergent* when the sum of the terms cannot be made to exceed numerically some finite quantity, and it is said to be *divergent* when the sum of the terms can be made numerically greater than any finite quantity.

For example, by actual division we obtain

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots$$

Now, if x is numerically less than 1, by taking a sufficiently large number of terms, the numerical value of the series can be made to approach the numerical value of $\frac{1}{1-x}$, to within any assignable limit. Thus, if $x = \frac{1}{2}$, then $\frac{1}{1-x} = 2$; and if we take four terms, the series equals 1.875. If we take five terms, it equals 1.9375; and, by taking terms enough, it may be made to differ from 2 by as small a quantity as we please; or, the series is actually equal to 2, when carried to infinity.

But if x is greater than 1, for instance, if it is equal to 2, the fraction equals -1; whereas, if we take four terms, the series is 15; if five terms, 31; and so on, where, by taking terms enough, the difference between the series and the fraction may be made greater than any finite quantity. In the former case the series is *convergent*, and in the latter *divergent*.

368. Infinite series may be developed by the common processes of *Division* and *Extraction* of *Roots*, and by other methods which it will now be our object to elucidate.

UNDETERMINED COEFFICIENTS.

369. A method of expanding algebraic fractions and other expressions into a series, simple in its principles, and general in its application, is based upon the following

THEOREM.

If the series $A + Bx + Cx^2 + Dx^3 + \ldots\ldots$ *is always equal to zero, whatever may be the value of* x, *the coefficients* A, B, C, D, ... *must each separately be equal to zero.*

For, since the series is to be zero, *whatever may be the value of* x, we may put $x = 0$; thus the series reduces to A, which must therefore itself be zero. Hence, removing this term, we have

$$Bx + Cx^2 + Dx^3 + \ldots\ldots$$

always zero; divide by x, and

$$B + Cx + Dx^2 + \ldots\ldots$$

is always zero. Hence, as before, we infer that $B = 0$. Proceeding in this way, the theorem is established.

Again, if the series

$$A + Bx + Cx^2 + Dx^3 + \ldots\ldots$$

and

$$A' + B'x + C'x^2 + D'x^3 + \ldots\ldots$$

are always equal, whatever may be the value of x, then

$$A - A' + (B - B')x + (C - C')x^2 + \ldots\ldots$$

is always zero, whatever may be the value of x; hence we infer that,

$$A - A' = 0, \quad B - B' = 0, \quad C - C' = 0 \ldots\ldots$$

That is,

The coefficients of like powers of x *in the two series are equal.*

The theorem here given is the *Principle of Undetermined Coefficients*, which are so called because, being assumed, the coefficients are at first unknown, or undetermined.

1. Let it be required to expand $\frac{1-x}{1+x}$ into an infinite series.

Assume $\frac{1-x}{1+x} = A + Bx + Cx^2 + Dx^3 + \ldots\ldots$ (1)

Clearing of fractions, and arranging the terms according to the powers of x, we have

$$1 - x = \begin{array}{l|l|l|l} A + B & x + C & x^2 + D & x^3 + \ldots\ldots \\ + A & + B & + C & \end{array}$$

But, by the principle of undetermined coefficients, the coefficients of like powers of x in the two members of the equation are equal. Therefore

$$\begin{array}{rll} A = 1, & \text{whence} & A = 1; \\ A + B = -1, & " & B = -2; \\ B + C = 0, & " & C = +2; \\ C + D = 0, & " & D = -2; \text{ etc.} \end{array}$$

Substituting these values in (1), we obtain

$$\frac{1-x}{1+x} = 1 - 2x + 2x^2 - 2x^3 + \ldots\ldots, \text{ Ans.}$$

which may be readily verified by division.

2. Let it be required to expand $\frac{a}{c+bx}$ into an infinite series.

Assume $\frac{a}{c+bx} = A + Bx + Cx^2 + Dx^3 + \ldots\ldots$

Clearing of fractions,

$$a = \begin{array}{l|l|l|l} Ac + Bc & x + Cc & x^2 + Dc & x^3 + \ldots\ldots \\ + Ab & + Bb & + Cb & \end{array}$$

Equating the coefficients of like powers of x, we obtain

$$A = \frac{a}{c}; \quad B = -\frac{ab}{c^2}; \quad C = +\frac{ab^2}{c^3}; \quad D = -\frac{ab^3}{c^4}; \text{ etc.}$$

Substituting, we have

$$\frac{a}{c+bx} = \frac{a}{c} - \frac{ab}{c^2}x + \frac{ab^2}{c^3}x^2 - \frac{ab^3}{c^4}x^3 + \ldots\ldots, \text{ Ans.}$$

3. Expand $\sqrt{1+x^2}$ into an infinite series.

Assume $\sqrt{1+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots$

Squaring both members,

$$1 + x^2 = \begin{array}{l|l|l|l|l} A^2 + AB & x + AC & x^2 + AD & x^3 + AE & x^4 \ldots\ldots \\ \quad + AB & \quad + B^2 & \quad + BC & \quad + BD & \\ & \quad + AC & \quad + BC & \quad + C^2 & \\ & & \quad + AD & \quad + BD & \\ & & & \quad + AE & \end{array}$$

Equating like coefficients,

$$\begin{aligned} A^2 &= 1, && \text{whence} && A &= 1; \\ 2AB &= 0, && \text{“} && B &= 0; \\ 2AC + B^2 &= 1, && \text{“} && C &= \tfrac{1}{2}; \\ 2AD + 2BC &= 0, && \text{“} && D &= 0; \\ 2AE + 2BD + C^2 &= 0, && \text{“} && E &= -\tfrac{1}{8}; \text{ etc.} \end{aligned}$$

Substituting, we have

$$\sqrt{1+x^2} = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \ldots\ldots, \text{ Ans.}$$

which may be verified by evolution.

Similarly may other like expressions be expanded. Hence, the

RULE.

Assume the given expression equal to a series with unknown coefficients, and clear the equation of fractions, or raise it to the power indicated.

Equate the coefficients of like powers of the unknown quantity in the two members of the equation, find the value of each, and substitute it in the assumed series.

EXAMPLES.

4. Expand $\frac{a}{b+x}$ into an infinite series.

Ans. $\frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4} + \ldots\ldots$

5. Expand $\frac{1}{1-2x+x^2}$ into an infinite series.

Ans. $1+2x+3x^2+4x^3+5x^4+\ldots\ldots$

6. Expand $(a^2-x^2)^{\frac{1}{2}}$ into an infinite series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\ldots\ldots$

Here, the operation will be shortened by assuming

$$(a^2-x^2)^{\frac{1}{2}}=A+Bx^2+Cx^4+Dx^6+Ex^8+\ldots\ldots$$

7. Expand $\frac{1+x}{1-x}$ into an infinite series.

Ans. $1+2x+2x^2+2x^3+\ldots\ldots$

8. Expand $\frac{1}{(a+2x)^3}$ into an infinite series.

Ans. $\frac{1}{a^3}-\frac{6x}{a^4}+\frac{24x^2}{a^5}-\frac{80x^3}{a^6}+\ldots\ldots$

9. Expand $(a+x)^{-2}$ into an infinite series.

Ans. $\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}+\ldots\ldots$

10. Expand $\frac{1}{3x-x^2}$ into an infinite series.

Ans. $\frac{x^{-1}}{3}+\frac{x^0}{9}+\frac{x}{27}+\frac{x^2}{81}+\ldots\ldots$

Here, taking the usual assumed series, and determining the coefficients, we obtain $1=0$, which is absurd. That form, therefore, is not applicable in the present case. By division, however, we discover that the first term of the required development is $\frac{1}{3x}$, or $\frac{x^{-1}}{3}$. We therefore assume

$$\frac{1}{3x-x^2}=Ax^{-1}+B+Cx+Dx^2+Ex^3+\ldots\ldots$$

11. Expand $\frac{1+2x}{1-x-x^2}$ into an infinite series.

Ans. $1+3x+4x^2+7x^3+11x^4+\ldots\ldots$

12. Expand $\frac{2}{3x^2-2x^3}$ into an infinite series.

Ans. $\frac{2x^{-2}}{3}+\frac{4x^{-1}}{9}+\frac{8}{27}+\frac{16x}{81}+\frac{32x^2}{243}+\ldots\ldots$

DECOMPOSITION OF RATIONAL FRACTIONS.

370. The principle of undetermined coefficients may be applied to the Decomposition of Fractions, whose denominators are expressed by factors, into *partial fractions* whose sum is equal the given fraction.

1. Let it be required to decompose $\frac{2a-x}{a^2-x^2}$.

Factoring a^2-x^2 gives $(a+x)(a-x)$.

Assume now $\frac{2a-x}{a^2-x^2}=\frac{A}{a+x}+\frac{B}{a-x}$

Clearing of fractions, and uniting,

$$2a-x=(A+B)a-(A-B)x$$

Equating the coefficients,

$$\left.\begin{matrix}A+B=2\\A-B=1\end{matrix}\right\}, \text{ whence } \left\{\begin{matrix}A=\frac{3}{2}\\B=\frac{1}{2}\end{matrix}\right.$$

Substituting,

$$\frac{2a-x}{a^2-x^2}=\frac{3}{2(a+x)}+\frac{1}{2(a-x)}, \text{ Ans.}$$

2. Decompose $\frac{3x-1}{x^2(x+1)^2}$ into its partial fractions.

Assume $\frac{3x-1}{x^2(x+1)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$.

Clearing of fractions,

$$3x-1 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2.$$

Equating the coefficients of like powers of x,

$$A = 5,\ B = -1,\ C = -5,\ D = -4.$$

Substituting,

$$\frac{3x-1}{x^2(x+1)^2} = \frac{5}{x} - \frac{1}{x^2} - \frac{5}{x+1} - \frac{4}{(x+1)^2}, \text{ Ans.}$$

Hence, the following

RULE.

Resolve the denominator of the given fraction into as many factors as possible. Take the several factors as denominators of partial fractions, and assume A, B, C, etc., respectively, as numerators. Find the value of these numerators by the principle of undetermined coefficients, and substitute them for those assumed.

EXAMPLES.

3. Resolve $\frac{3x-5}{x^2-13x+40}$ into its partial fractions.

Ans. $\frac{19}{3(x-8)} - \frac{10}{3(x-5)}$.

4. Resolve $\frac{1+x^2}{x-x^3}$ into its partial fractions.

Ans. $\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x}$.

5. Resolve $\frac{x^2}{(x^2-1)(x-2)}$ into its partial fractions.

Ans. $\frac{4}{3(x-2)} - \frac{1}{2(x-1)} + \frac{1}{6(x+1)}$.

6. Resolve $\frac{3x+2}{x(x+1)^3}$ into its partial fractions.

Ans. $\frac{2}{x} - \frac{2}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}$.

BINOMIAL THEOREM. ANY EXPONENT.

371. We have seen (Art. 356) that when n is a positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \ldots\ldots$$

We shall now show that this formula holds when n has any value, positive or negative, integral or fractional; that is, we shall prove the Binomial Theorem for *any* exponent.

Let n be negative, or $= -n$; then, by actual division,

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \frac{1}{1+nx+\ldots\ldots} = 1 - nx + \ldots\ldots$$

and the exponents increase as before.

Let n be a fraction, and be denoted by $\frac{p}{q}$; then, since any power or root of 1 is 1, we assume

$$(1+x)^{\frac{p}{q}} = 1 + Ax + Bx^2 + Cx^3 + \ldots\ldots$$

whence,

$$(1+x)^p = (1 + Ax + Bx^2 + Cx^3 + \ldots\ldots)^q$$

or, $$1 + px + \ldots\ldots = 1 + qAx + \ldots\ldots$$

Equating the coefficients (Art. 369), we obtain

$$p = qA; \text{ whence } \frac{p}{q} = A;$$

and $$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + Bx^2 + Cx^3 + \ldots\ldots$$

Hence, whether the exponent n be positive or negative integral or fractional, the form of the expansion will be

$$(1+x)^n = 1 + nx + Bx^2 + Cx^3 + Dx^4 + \ldots\ldots \quad (1)$$

where the coefficient of the second term is the same as the exponent of the power. Thus two terms of the series are determined.

To find now the other coefficients, we put $x+z$ for x in (1), and, regarding $(x+z)$ as one term, we shall have

$$[1+(x+z)]^n = 1+n(x+z)+B(x+z)^2+C(x+z)^3+\ldots\ldots$$
$$= 1+nx+Bx^2+Cx^3+Dx^4+\ldots\ldots$$
$$+(n+2Bx+3Cx^2+4Dx^3+\ldots)z+\ldots \quad (2)$$

which is true for all values of x and z.

Regarding $(1+x)$ as one term, we shall have

$$[(1+x)+z]^n = (1+x)^n+n(1+x)^{n-1}z+\ldots\ldots \quad (3)$$

But (Art. 369), the coefficients of z in each must be the same, and

$$n(1+x)^{n-1} = n+2Bx+3Cx^2+4Dx^3+\ldots\ldots$$

for all values of x.

Multiplying by $1+x$,

$$n(1+x)^n = \begin{array}{l|l|l|l} n+2B & x+3C & x^2+4D & x^3+\ldots\ldots \\ +n & +2B & +3C & \end{array}$$

Equating the coefficients of the same powers of x, we have

$$2B+n=n^2,\quad 3C+2B=nB,\quad \text{and}\quad 4D+3B=nC.$$

$2B=n(n-1)$; whence $B=\frac{n(n-1)}{1\cdot 2}$;

$3C=B(n-2)$; whence $C=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$;

$4D=C(n-3)$; whence $D=\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$;

etc.; whence,

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\ldots\ldots \quad (A)$$

and the Binomial Theorem is established in the most general form.

372. By substituting $\frac{x}{a}$ for x, and multiplying both members by a^n (Art. 358), (A) becomes

$$\left.\begin{aligned}(a+x)^n = a^n\left(1+\tfrac{x}{a}\right)^n &= a^n\left(1+n\tfrac{x}{a}+\tfrac{n(n-1)}{1.2}\tfrac{x^2}{a^2}+\tfrac{n(n-1)(n-2)}{1.2.3}\tfrac{x^3}{a^3}+\ldots\right)\\ &= a^n+na^{n-1}x+\tfrac{n(n-1)}{1.2}a^{n-2}x^2+\tfrac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3+\ldots.\end{aligned}\right\} \quad (B)$$

Any power or root of any binomial may be developed by substituting in (A) or (B) the given values of n, x, and a. It may sometimes be convenient, however, when n is negative or fractional, to make use of one of the following reduced forms.

$$(1+x)^{-n} = 1-nx+\frac{n(n+1)}{1.2}x^2-\frac{n(n+1)(n+2)}{1.2.3}x^3+\ldots. \quad (C)$$

$$(1+x)^{\frac{p}{q}} = 1+\frac{p}{q}x+\frac{p(p-q)}{1.2.q^2}x^2+\frac{p(p-q)(p-2q)}{1.2.3.q^3}x^3+\ldots. \quad (D)$$

$$(1+x)^{-\frac{p}{q}} = 1-\frac{p}{q}x+\frac{p(p+q)}{1.2.q^2}x^2-\frac{p(p+q)(p+2q)}{1.2.3.q^3}x^3+\ldots. \quad (E)$$

373. If x is negative, all odd powers of x will be negative, and therefore the even terms of the above series will change their signs.

374. It has been shown (Art. 360), if n be a positive integer, that the expansion will terminate with the $(n+1)$th term.

If n be either negative or fractional, it is manifest, since no one of the factors of the rth term can ever become 0, the series never terminates; consequently the development furnishes an infinite series.

It will be seen that each coefficient may be obtained from the preceding one, by multiplying it by certain factors, easily determined.

Examples.

1. Expand $(1+x)^{-2}$ into an infinite series.

Here, $n = -2$; hence, by formula (A),

$$(1+x)^{-2} = 1 + \frac{-2}{1}x + \frac{-2(-2-1)}{1\,.\,2}x^2 + \frac{-2(-2-1)(-2-2)}{1\,.\,2\,.\,3}x^3 + \ldots.$$

$$= 1 - \frac{2}{1}x + \frac{2\,.\,3}{1\,.\,2}x^2 - \frac{2\,.\,3\,.\,4}{1\,.\,2\,.\,3}x^3 + \ldots\ldots$$

$$= 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots, \text{ Ans.}$$

2. Expand $(1-x)^{-\frac{2}{3}}$ into an infinite series.

Here, $n = -\frac{2}{3}$; hence, by formula (E) and Art. 373,

$$(1-x)^{-\frac{2}{3}} = 1 + \tfrac{2}{3}x + \frac{2\,(2+3)}{1\,.\,2\,.\,3^2}x^2 + \frac{2(2+3)(2+2\,.\,3)}{1\,.\,2\,.\,3\,.\,3^3}x^3 + \ldots..$$

$$= 1 + \tfrac{2}{3}x + \frac{2\,.\,5}{1\,.\,2\,.\,9}x^2 + \frac{2\,.\,5\,.\,8}{1\,.\,2\,.\,3\,.\,27}x^3 + \ldots..$$

$$= 1 + \tfrac{2}{3}x + \tfrac{5}{9}x^2 + \tfrac{40}{81}x^3 + \ldots\ldots, \text{ Ans.}$$

3. Expand $(a+x)^{-2}$ into an infinite series.

Here, the expansion may be made directly by formula (B), or, after the expression is transformed into $\frac{1}{a^2}\left(1+\frac{x}{a}\right)^{-2}$, (C) may be used.

$$(a+x)^{-2} = a^{-2} - 2a^{-3}x + 3a^{-4}x^2 - 4a^{-5}x^3 + \ldots..$$

$$= \frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{4x^2}{a^5} + \ldots\ldots, \text{ Ans.}$$

4. Expand $(1+2x)^{\frac{1}{2}}$ into an infinite series.

Ans. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \ldots\ldots$

5. Expand $\frac{1}{1-x}$, or $(1-x)^{-1}$, into an infinite series.

Ans. $1 + x + x^2 + x^3 + x^4 + \ldots\ldots$

6. Convert $\frac{1}{(1+x)^{\frac{1}{5}}}$ into an infinite series.

Ans. $1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \ldots\ldots$

7. Convert $2\,(c+x)^{-2}$ into an infinite series.

Ans. $\frac{2}{c^2} - \frac{4x}{c^3} + \frac{6x^2}{c^4} - \frac{8x^3}{c^5} + \ldots\ldots$

8. Convert $\frac{1}{\sqrt{1+x^2}}$ into an infinite series.

Ans. $1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{35}{128}x^8 - \ldots\ldots$

9. Convert $x^2\,(x^2-y)^{-\frac{1}{2}}$ into an infinite series.

Ans. $x + \frac{y}{2x} + \frac{3y^2}{8x^3} + \frac{5y^3}{16x^5} + \frac{35y^4}{128x^7} + \ldots\ldots$

10. Convert $(a^2-x^2)^{-\frac{3}{4}}$ into an infinite series.

Ans. $a^{-\frac{3}{2}} + \frac{3}{4}a^{-\frac{7}{2}}x^2 + \frac{21}{32}a^{-\frac{11}{2}}x^4 + \frac{77}{128}a^{-\frac{15}{2}}x^6 + \ldots.$

11. Convert $\frac{m}{\sqrt{b^2+c^4}}$ into an infinite series.

Ans. $\frac{m}{b}\left(1 - \frac{c^4}{2b^2} + \frac{3c^8}{8b^4} - \frac{5c^{12}}{16b^6} + \frac{35c^{16}}{128b^6} - \ldots\ldots\right).$

375. The binomial theorem may be employed to find the approximate roots of numbers.

Let N be a number whose nth root is required, and suppose $N = a^n \pm b$; then

$$N^{\frac{1}{n}} = (a^n \pm b)^{\frac{1}{n}} = a\left(1 \pm \frac{b}{a^n}\right)^{\frac{1}{n}}.$$

If, now, $\frac{b}{a^n}$ be a small fraction, the terms in the expansion of $\left(1 \pm \frac{b}{a^n}\right)^{\frac{1}{n}}$ converge rapidly, and we obtain an approximate value of $\left(1 \pm \frac{b}{a^n}\right)^{\frac{1}{n}}$, and therefore of $N^{\frac{1}{n}}$, by taking the sum of a few terms of the development. But if the expansion converges slowly, it will require the sum of many terms to insure a considerable degree of accuracy.

12. Find the approximate cube root of 31.

Here, $\sqrt[3]{31} = \sqrt[3]{3^3 + 4} = 3\,(1 + \frac{4}{27})^{\frac{1}{3}}$; and

$$3\,(1 + \tfrac{4}{27})^{\frac{1}{3}} = 3 + \tfrac{4}{27} - \tfrac{16}{2187} + \tfrac{320}{531441} - \tfrac{2560}{43046721} + \ldots\ldots$$

The value of the first term is		$+3.00000$
"	second "	$+0.14815$
"	third "	-0.00731
"	fourth "	$+0.00060$
"	fifth "	-0.00006
The sum of the five terms		$=3.14138$

which is the approximate value of $\sqrt[3]{31}$ to the fifth decimal place, as may be verified by evolution.

13. Find the approximate value of $\sqrt[3]{9}$ to five decimal places. Ans. 2.08008.

14. Find the approximate value of $\sqrt[5]{39}$ to four decimal places. Ans. 2.0807.

15. Find the approximate fifth root of 260, or $(3^5 + 17)$, to the fifth decimal place. Ans. 3.04085.

16. Find the approximate value of $\sqrt[7]{108}$, or $\sqrt[7]{128 - 20}$, to the fifth decimal place. Ans. 1.95204.

REVERSION OF SERIES.

376. The Reversion of a Series is the process of finding an expression of the value of its unknown quantity in terms of another unknown quantity.

Let there be given $y = ax + bx^2 + cx^3 + \ldots\ldots$, to revert the series, or to find the value of x in terms of y.

In order to express the value of x in terms of y, by the method of undetermined coefficients, assume

$$x = Ay + By^2 + Cy^3 + Dy^4 + \ldots\ldots$$

Substituting this value of x in the equation

$$y = ax + bx^2 + cx^3 + dx^4 + \ldots\ldots,$$

it becomes

$$y = aAy + aB \left| \begin{array}{l} +bA^2 \end{array} \right| y^2 + aC \left| \begin{array}{l} +2bAB \\ +cA^3 \end{array} \right| y^3 + aD \left| \begin{array}{l} +2bAC \\ +bB^2 \\ +3cA^2B \\ +dA^4 \end{array} \right| y^4 + \ldots\ldots$$

Equating the like coefficients of y,

$$aA = 1; \text{ whence } A = \frac{1}{a}.$$

$$aB + bA^2 = 0; \quad \text{“} \quad B = -\frac{b}{a^3}.$$

$$aC + 2bAB + cA^3 = 0; \quad \text{“} \quad C = \frac{2b^2 - ac}{a^5}.$$

$$aD + 2bAC + bB^2 + 3cA^2B + dA^4 = 0; \quad \text{“} \quad D = -\frac{5b^3 - 5abc + a^2d}{a^7}.$$

Hence,

$$x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 - \frac{5b^3 - 5abc + a^2d}{a^7}y^4 + \ldots\ldots \quad (A)$$

If the even powers of x be wanting in the series, thus,

$$y = ax + bx^3 + cx^5 + \ldots\ldots,$$

we assume for x a series containing the odd powers of y, and obtain

$$x = \frac{1}{a}y - \frac{b}{a^4}y^3 + \frac{3b^2 - ac}{a^7}y^5 - \frac{12b^3 - 8abc + a^2d}{a^{10}}y^7 + \ldots\ldots \quad (B)$$

In like manner, by application of the principle of undetermined coefficients, or by the direct application of (A) and (B) as formulas, other series of similar form may be reverted.

EXAMPLES.

1. Revert the series $y = x + x^2 + x^3 + x^4 + \ldots\ldots$

Ans. $x = y - y^2 + y^3 - y^4 + \ldots\ldots$

2. Revert the series $y = 2x + 3x^3 + 4x^5 + 5x^7 + \ldots\ldots$

Ans. $x = \frac{1}{2}y - \frac{3}{16}y^3 + \frac{19}{128}y^5 - \frac{1525}{1024}y^7 + \ldots\ldots$

3. Revert the series $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$,

or $y - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$

Ans. $x = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 + \ldots\ldots$

4. Revert the series $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\ldots$

Ans. $x = y + \frac{1}{3}y^3 + \frac{2}{15}y^5 + \frac{17}{315}y^7 + \ldots\ldots$

5. Revert the series $ax + bx^2 + cx^3 + \ldots\ldots = my + ny^2 + py^3 + \ldots\ldots$

Ans. $y = \frac{a}{m}x + \frac{bm^2 - a^2n}{m^3}x^2 + \frac{cm^4 - a^3mp - 2an(bm^2 - a^2n)}{m^5}x^3 + \ldots\ldots$

SUMMATION OF INFINITE SERIES.

377. The Summation of a Series is the process of finding a finite expression equivalent to the series.

Different series require different methods of summing, according to the nature of the series, or the law of its formation. Methods of summing arithmetical and geometrical series have already been given (Arts. 336, 341). Methods applicable to other series will now be treated.

RECURRING SERIES.

378. A Recurring Series is one in which a certain number of consecutive terms, taken in any part of the series, bear a uniform relation to the term immediately following. Thus,

$$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$$

is a recurring series, in which each term after the first two is equal to the product of the first preceding term

by $2x$, plus the product of the second preceding term by $-x^2$.

These constant multipliers, $2x - x^2$, are called the *scale of relation* of the terms, and their coefficients, $2 - 1$, constitute the scale of relation of the coefficients of the series.

379. A recurring series is said to be of the *first order* when each term, commencing with the second, depends upon the *one* immediately preceding; of the *second order*, when each term, commencing with the third, depends upon the *two* preceding; and so on.

If the series is of the first order, the scale of relation will consist of one term; if of the second order, it will consist of two terms; and, in general, the order and the number of terms in the scale will correspond.

380. *To determine the scale of relation of a recurring series.*

1. If the series is of the first order, it is a simple geometrical progression, and the scale of relation of the terms is the ratio of any two consecutive terms.

2. If the series is of the second order, let a, b, c, d, ... represent the consecutive coefficients of the series, and $p + q$ their scale of relation. Then we have

$$\left.\begin{aligned} c &= pb + qa \\ d &= pc + qb \end{aligned}\right\} \quad \text{(A)}$$

to determine p and q; and reducing, we obtain

$$p = \frac{ad - bc}{ac - b^2}, \text{ and } q = \frac{c^2 - bd}{ac - b^2}.$$

3. If the series is of the third order, let a, b, c, d, e, f, ... represent the consecutive coefficients of the series, and $p + q + r$ their scale of relation. Then we have

$$\left.\begin{aligned} d &= pc + qb + ra \\ e &= pd + qc + rb \\ f &= pe + qd + rc \end{aligned}\right\} \quad (B)$$

from which we can find p, q, and r.

To ascertain the order of the series, we may first make trial of a scale of two terms, and if the result does not correspond with the series, we may try three terms, four terms, and so on, till the true scale is found. If we assume the series to be of too high an order, one or more terms of the scale will be found equal to 0, and the others will be the true scale.

381. *To find the sum of a recurring series whose scale of relation is known.*

Let $$a + bx + cx^2 + dx^3 + ex^4 + \ldots\ldots$$

be a recurring series of the second order. Denoting the sum of n terms of the series by s, we have

$$\begin{aligned} S &= a + bx + cx^2 + dx^3 + \ldots\ldots + lx^{n-1} \\ pSx &= pax + pbx^2 + pcx^3 + \ldots\ldots + pkx^{n-1} + plx^n \\ qSx^2 &= \qquad qax^2 + qbx^3 + \ldots\ldots + qjx^{n-1} + qkx^n + qlx^{n+1}. \end{aligned}$$

Subtracting the last two equations from the first, we have

$$S - pSx - qSx^2 = a + bx - pax - plx^n - qkx^n - qlx^{n+1};$$

for all the other terms of the second member disappear by means of the relation expressed in equations (A).

Therefore we have

$$S = \frac{a + (b - pa)x - (pl + qk)x^n - qlx^{n+1}}{1 - px - qx^2}, \qquad (C)$$

the formula for finding the sum of n terms of a recurring series of the second order.

But if n becomes infinite, and the series is converging, then the terms which involve x^n and x^{n+1} must become 0, and we have

$$S = \frac{a + (b - pa)\,x}{1 - px - qx^2}, \qquad \text{(D)}$$

the formula for finding the sum of an infinite recurring series of the second order.

If $q = 0$, then $b = pa$, and we have

$$S = \frac{a}{1 - px}, \qquad \text{(E)}$$

the formula for finding the sum of an infinite recurring series of the first order (Art. 342).

In like manner we obtain

$$S = \frac{a + (b - pa)\,x + (c - pb - qa)\,x^2}{1 - px - qx^2 - rx^3}, \qquad \text{(F)}$$

the formula for the summation of an infinite recurring series of the third order.

A recurring series, like other infinite series, originates from an irreducible fraction, called the *generating fraction.*

The summation of a recurring series, therefore, reproduces the fraction. From formulas (D), (E), and (F), we infer that the number of terms in the numerator of that fraction is the same as the order of the series, and in the denominator it is one greater.

Examples.

1. Find the sum of $1 + 2x + 8x^2 + 28x^3 + 100x^4 + \ldots\ldots$

Here, $a = 1$, $b = 2$, $c = 8$, $d = 28$, $e = 100$, etc.

Substituting these values in formulas (A), we obtain

$$p = 3, \quad \text{and} \quad q = 2;$$

whence, by formula (D), we find

$$S = \frac{1 + (2 - 3)\,x}{1 - 3x - 2x^2} = \frac{1 - x}{1 - 3x - 2x^2}, \quad \text{Ans.}$$

2. Find the sum of $1 + 2x + 3x^2 + 5x^3 + 8x^4 + \ldots\ldots$

Ans. $\frac{1+x}{1-x-x^2}$.

3. Find the sum of $\frac{a}{b} - \frac{ac}{b^2}x + \frac{ac^2}{b^3}x^2 - \frac{ac^3}{b^4}x^3 + \ldots\ldots$

Ans. $\frac{a}{b+cx}$.

4. Find the sum of $4 + 9x + 21x^2 + 51x^3 + \ldots\ldots$

Ans. $\frac{4-11x}{1-5x+6x^2}$.

5. Find the sum of $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots$

Ans. $\frac{1+x}{(1-x)^2}$.

6. Find the expression from which is derivable $2 - a + 2a^2 - 5a^3 + 10a^4 - 17a^5 + \ldots\ldots$

Ans. $\frac{2+5a+5a^2}{(1+a)^3}$.

7. Find the expression from which is derivable $3 + 5x + 7x^2 + 13x^3 + 23x^4 + 45x^5 + \ldots\ldots$

Ans. $\frac{3-x-6x^2}{1-2x-x^2+2x^3}$.

DIFFERENTIAL METHOD.

382. The DIFFERENTIAL METHOD is the process of finding any term, or the sum of any number of terms, of a regular series, by means of the successive differences of its terms.

383. If, in any series, we take the first term from the second, the second from the third, the third from the fourth, and so on, the remainders will form a new series called the *first order of differences.*

If the differences be taken in this new series in like manner, we obtain the *second order of differences*, and so on.

Thus, if the given series is

$$1,\ 8,\ 27,\ 64,\ 125, \ldots\ldots,$$

the successive order of differences will be as follows: —

1st order,	7, 19, 37, 61, ...
2d order,	12, 18, 24, ...
3d order,	6, 6, ...
4th order,	0 . . . . .

Hence, in this case, there will be only three orders of differences.

384. *To find any term of a series.*

Let the series be

$$a,\ b,\ c,\ d,\ e, \ldots\ldots.$$

then, the first order of differences is

$$b-a,\ \ c-b,\ \ d-c,\ \ e-d, \ldots\ldots$$

the second order of differences is

$$c-2b+a,\ \ d-2c+b,\ \ e-2d+c, \ldots\ldots$$

the third order of differences is

$$d-3c+3b-a,\ \ e-3d+3c-b, \ldots\ldots$$

the fourth order of differences is

$$e-4d+6c-4b+a, \ldots\ldots$$

where each difference, although a compound quantity, is called a *term*.

Let, now, $d_1, d_2, d_3, d_4, \ldots\ldots$ represent the first terms of the several orders of differences. Then,

$d_1 = b-a;$	whence	$b = a+d_1;$
$d_2 = c-2b+a;$	"	$c = a+2d_1+d_2;$
$d_3 = d-3c+3b-a;$	"	$d = a+3d_1+3d_2+d_3;$
$d_4 = e-4d+6c-4b+a;$	"	$e = a+4d_1+6d_2+4d_3+d_4;$
etc.		etc.

where the coefficients of the value of b, the *second* term in the given series, are the coefficients of the *first* power of a binomial; the coefficients of the value of c, the *third* term, are the coefficients of the *second* power of a binomial; and so on. Hence, we infer that the coefficients of the nth term of the series are the coefficients of the $(n-1)$th power of a binomial; whence, representing the nth term of the series by T_n, we shall have

$$T_n = a + (n-1)d_1 + \frac{(n-1)(n-2)}{1 \cdot 2} d_2 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} d_3 + \ldots \quad (A)$$

When the differences at last become 0, the value of the nth term can be obtained exactly; but, in other cases, the result will be merely an approximate value.

385. *To find the sum of any number of terms of a series.* Let the series be

$$a, \quad b, \quad c, \quad d, \quad e, \ldots\ldots \quad (1)$$

Assume the series

$$0, \quad 0 + a, \quad a + b, \quad a + b + c, \quad a + b + c + d, \ldots\ldots \quad (2)$$

where the $(n+1)$th term is obviously equal to the sum of n terms in the proposed series, and the first order of differences is equal to series (1). Hence, the $(n+1)$th order of differences in series (2) is the nth order in series (1). If, then, we substitute in formula (A), 0 for a, $n+1$ for n, a for d_1, d_1 for d_2, etc., and denote the sum of n terms of the proposed series by S_n, we shall obtain

$$S_n = na + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \ldots\ldots \quad (B)$$

EXAMPLES.

1. Find the 12th term of the series

$$2, \quad 6, \quad 12, \quad 20, \quad 30, \ldots\ldots$$

OPERATION.

$$2,\ 6,\ 12,\ 20,\ 30, \ldots\ldots$$
$$4,\ 6,\ 8,\ 10, \ldots\ldots$$
$$2,\ 2,\ 2, \ldots\ldots$$
$$0,\ 0, \ldots\ldots$$

Hence $a = 2$, $d_1 = 4$, $d_2 = 2$, and $n = 12$.

Then, by substituting in formula (A), we have for the 12th term

$$2 + (12-1)\,4 + \frac{(12-1)\,(12-2)}{1 \cdot 2}\,2 = 2 + 44 + 110 = 156, \text{ Ans.}$$

2. Find the sum of 8 terms of the series 2, 5, 10, 17, ...

Here, we find $a = 2$, $d_1 = 3$, $d_2 = 2$, and $n = 8$.

Then, by substituting in formula (B), we have

$$S_8 = 8 \times 2 + \frac{8\,(8-1)}{1 \cdot 2}\,3 + \frac{8\,(8-1)\,(8-2)}{1 \cdot 2 \cdot 3}\,2$$
$$= 16 + 84 + 112 = 212, \text{ Ans.}$$

3. Find the first term of the fifth order of differences of the series 6, 9, 17, 35, 63, 99, ... Ans. 3.

4. Find the first term of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, ...

Ans. -14.

5. Find the seventh term of the series 3, 5, 8, 12, 17, ...

Ans. 30.

6. Sum the series 1, 4, 10, 20, 35, ... to the 12th term. Ans. 1365.

7. Sum the series 1, 2, 3, 4, 5, ... to the 100th term. Ans. 5050.

8. Find the 15th term of the series $1^2, 2^2, 3^2, 4^2, \ldots\ldots$

Ans. 225.

9. Sum $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots\ldots$ to the nth term.

Ans. $\frac{1}{4}(n^4 + 2n^3 + n^2)$.

10. Sum $1 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 + \ldots\ldots$ to the nth term. Ans, $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

11. If shot be piled in the shape of a pyramid, with a triangular base, each side of which exhibits 9 shot, find the number contained in the pile. Ans. 165.

12. If shot be piled in the shape of a pyramid, with a square base, each side of which exhibits 25 shot, find the number contained in the pile. Ans. 5525.

INTERPOLATION.

386. INTERPOLATION is the process of introducing between terms of a series other terms conforming to the law of the series.

Its usual application is in finding *intermediate* numbers between those given in Mathematical Tables, which may be regarded as a series of equidistant terms.

387. The interpolation of any intermediate term in a series, is essentially finding the nth term of the series, by the differential method (Art. 384). Thus,

Let t represent the term to be interpolated in a series of equidistant terms, and p the distance the term t is removed from the first term, a, that is, $p = n - 1$.

Then, by substituting p for $n - 1$ in formula (A), Art. 384, we have the formula of interpolation,

$$t = a + pd_1 + \frac{p(p-1)}{1.2} d_2 + \frac{p(p-1)(p-2)}{1.2.3} d_3 + \ldots\ldots$$

EXAMPLES.

1. In the series $\frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \ldots\ldots$, find the middle term between $\frac{1}{15}$ and $\frac{1}{16}$.

Here, the first differences of the denominators are

$$1, \quad 1, \quad 1, \quad 1, \ldots\ldots$$

The second differences are

$$0, \quad 0, \quad 0, \ldots\ldots$$

whence, $d_1 = 1$, and $d_2 = 0$.

Since each interval between the terms is to be reckoned as unity, the distance from the first term to the required middle term is $2\frac{1}{2}$ intervals, or $p = 2\frac{1}{2}$.

Make $a = 13$, the denominator of the first term; then, by the preceding formula,

$$13 + 2\tfrac{1}{2} \times 1 = 15\tfrac{1}{2}, \quad \text{and} \quad 1 \div 15\tfrac{1}{2} = \tfrac{2}{31},$$

or the proposed middle term.

2. Given the square root of $94 = 9.69536$, of $95 = 9.74679$, and of $96 = 9.79796$, to find the square root of $94\frac{1}{4}$.

Here, the first differences are

$$.05143, \quad .05117;$$

and the second difference is

$$-.00026;$$

whence, $d_1 = .05143$, and $d_2 = -.00026$.

Since the distance is $\frac{1}{4}$ of the first interval, $p = \frac{1}{4}$.

Hence, $\sqrt{94\frac{1}{4}} = 9.69536 + \frac{1}{4}(.05143) - \frac{3}{32}(-.00026)$
$= 9.69536 + .01286 + .00002 = 9.70824$, Ans.

3. Given the cube root of $64 = 4$, of $65 = 4.0207$, of $66 = 4.0412$, and of $67 = 4.0615$, to find the cube root of 66.5. Ans. 4.0514.

4. Required the number of miles in a degree of longitude in latitude 42° 30′, the length of a degree of longitude

in latitude 41° being 45.28 miles, in latitude 42° being 44.59 miles, in latitude 43°, 43.88 miles, and in latitude 44°, 43.16 miles. Ans. 44.24 miles.

5. Given the cube root of $45 = 3.556893$, of $47 = 3.608826$, of $49 = 3.659306$, and of $51 = 3.708430$, to find the cube root of 48. Ans. 3.634241.

6. If the amount of \$1 at 7 per cent compound interest for 2 years is \$1.145, for 3 years is \$1.225, for 4 years \$1.311, and for 5 years \$1.403, what is the amount for 4 years and 6 months? Ans. \$1.357.

LOGARITHMS.

388. The LOGARITHM of a number is the exponent of the power to which some fixed number, called the *base*, must be raised, to produce the given number. Thus, suppose

$$a^x = n,$$

then x is the logarithm of n to the base a, and may be written

$$x = \log_a n,$$

where $\log_a$ is read logarithm to base a. The subscript is usually omitted when the base is readily understood.

389. If, in the preceding equation, a remaining fixed, n be supposed to assume in succession all positive values, the corresponding values of x, taken together, will constitute a *system* of logarithms.

But, since a may be made any positive number greater than unity, there may be *different* systems of logarithms.

THE COMMON SYSTEM.

390. The base, in the common system of logarithms, is 10. Hence, since

$10^0 = 1,$	0	is the	logarithm	of 1;
$10^1 = 10,$	1	"	"	10;
$10^2 = 100,$	2	"	"	100;
$10^3 = 1000,$	3	"	"	1000;
etc.				etc.

It thus appears that, in the common system, the logarithm of every number between 1 and 10 is some number between 0 and 1; that is, a proper fraction. The logarithm of every number between 10 and 100 is some number between 1 and 2; that is, 1 plus a fraction. The logarithm of every number between 100 and 1000 is some number between 2 and 3; that is, 2 plus a fraction; and so on.

391. By means of *negative* exponents the application of logarithms may be extended to numbers less than 1. Thus, since

$10^{-1} = 0.1,$	-1	is the	logarithm	of 0.1;
$10^{-2} = 0.01,$	-2	"	"	0.01;
$10^{-3} = 0.001,$	-3	"	"	0.001;
etc.				etc.

From this it appears that the logarithm of every number between 1 and 0.1 is some number between 0 and -1; that is, -1 plus a fraction. The logarithm of every number between 0.1 and 0.01 is some number between -1 and -2; that is, -2 plus a fraction. The logarithm of every number between 0.01 and 0.001 is some number between -2 and -3; that is, -3 plus a fraction; and so on.

392. The logarithms of all numbers which are not exact powers of 10, being incommensurable with those

numbers, can have their values only approximately obtained, and therefore are usually expressed by means of a decimal fraction.

393. The CHARACTERISTIC of a logarithm is its *integral* part.

The *decimal* part is sometimes called the *mantissa.*

394. *The characteristic of the logarithm of any* INTEGER, *or* MIXED NUMBER, *is positive, and equal to the number of integral places less one.*

This is evident from Art. 390.

395. *The characteristic of the logarithm of any* DECIMAL FRACTION *is negative, and numerically one more than the number of ciphers between the decimal point and the first significant figure.*

This is evident from Art. 391.

PROPERTIES OF LOGARITHMS.

396. *The logarithm of* 1 *in any system is equal to* 0.

For, in the equation $a^x = m$, where x is the logarithm of m, whatever be the value of the base a, make $x = 0$, and we have

$$a^0 = m = 1, \quad \text{or} \quad \log 1 = 0.$$

397. *The logarithm of the base itself, in any system, is* 1.

For, make $x = 1$ in the equation $a^x = m$, and we have

$$a^1 = m = a, \quad \text{or} \quad \log a = 1.$$

398. *A negative quantity has no real logarithm.*

For, since the base a is positive, all its powers must be positive (Art. 202).

399. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

For, let $m = a^x$, and $n = a^y$;

then, multiplying these equations, member by member, we have

$$mn = a^x a^y = a^{x+y}.$$

Therefore,

$$\log (mn) = x + y = \log m + \log n.$$

400. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by that of the divisor.*

For, let $m = a^x$, and $n = a^y$;

then, dividing the first equation by the second, member by member, we have

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}.$$

Therefore,

$$\log \left(\frac{m}{n}\right) = x - y = \log m - \log n.$$

401. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, let $m = a^x$;

then, raising both members to the rth power, we have

$$m^r = (a^x)^r = a^{xr}.$$

Therefore, $\log (m^r) = xr = r \log m.$

402. *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

For, let $m = a^x$;

then, extracting the rth root of both members, we have

$$\sqrt[r]{m} = \sqrt[r]{a^x} = a^{\frac{x}{r}}.$$

Therefore, $\log (\sqrt[r]{m}) = \frac{x}{r} = \frac{\log m}{r}.$

403. *The arithmetical mean of the logarithms of two numbers is the logarithm of the geometrical mean of those numbers.*

For, let $a^x = m$, and $a^y = n$;

then $a^{x+y} = mn$,

and $a^{\frac{x+y}{2}} = \sqrt{mn}$,

or, $\frac{\log m + \log n}{2} = \log \sqrt{mn}$. (Arts. 337, 343.)

404. From the foregoing, it is evident that multiplication, division, involution, and evolution may be performed by means of logarithms; hence they are often of great practical utility in shortening numerical calculations.

In computations by logarithms, negative quantities are used as if they were positive (Art. 398), the *sign* of the result being determined irrespective of the logarithmic work.

COMPUTATION OF LOGARITHMS.

405. The properties just demonstrated are true of every system of logarithms; but their application to numerical calculations supposes the construction of a logarithmic table.

A table of logarithms may be computed by means of Art. 403; but the following is a more convenient method.

Let us resume the equation

$$a^x = n, \tag{1}$$

where x is the logarithm of n, to the base a.

Assume $a = 1 + m$, and $n = 1 + p$,

then $$(1 + m)^x = 1 + p. \tag{2}$$

Raising both members of this equation to the nth power, we have

$$(1+m)^{nx} = (1+p)^n,$$

for every value of n.

Expanding both members of the last equation by the Binomial Formula, we obtain

$$1+nxm+\frac{nx(nx-1)}{1.2}m^2+\frac{nx(nx-1)(nx-2)}{1.2.3}m^3+\ldots\ldots$$
$$=1+np+\frac{n(n-1)}{1.2}p^2+\frac{n(n-1)(n-2)}{1.2.3}p^3+\ldots\ldots$$

Omitting 1 from both members, dividing by n, and factoring the first member, we find

$$x\left(m+\frac{nx-1}{1.2}m^2+\frac{(nx-1)(nx-2)}{1.2.3}m^3+\ldots\ldots\right)$$
$$=p+\frac{n-1}{1.2}p^2+\frac{(n-1)(n-2)}{1.2.3}p^3+\ldots\ldots$$

Make $n=0$, and the last equation reduces to

$$x\left(m-\frac{m^2}{2}+\frac{m^3}{3}-\ldots\ldots\right)=p-\frac{p^2}{2}+\frac{p^3}{3}-\ldots\ldots \quad (3)$$

But, by equation (2),

$$x=\log(1+p).$$

Hence, if we make the reciprocal of the other factor of the same member equal to M, equation (3) becomes

$$\log(1+p)=\log n=M\left(p-\frac{p^2}{2}+\frac{p^3}{3}-\ldots\ldots\right), \quad (A)$$

which is an expression for the logarithm of any number n. This expression is composed of two factors; the one within the parenthesis, dependent upon the number, and the other, M, dependent upon the base of the system.

The factor dependent upon the base is called the *modulus* of the system of logarithms.

THE NAPIERIAN SYSTEM.

406. The *Napierian system*, or that invented by Baron Napier, has the modulus arbitrarily assumed as unity, or

$$M = 1.$$

Denoting the logarithms in the Napierian system by $\log_e$, we have, from formula (A),

$$\log_e (1+p) = p - \frac{p^2}{2} + \frac{p^3}{3} - \ldots\ldots \qquad (B)$$

This series may be applied to the calculation of Napierian logarithms, if p is a proper fraction, but unless p be very small, it converges so slowly that a large number of terms are required to obtain the accuracy desirable; and if p be greater than 1, the series becomes divergent, and consequently altogether unsuitable.

It can, however, be converted into another more convenient series. Thus,

$$\log_e (1+p) = p - \frac{p^2}{2} + \frac{p^3}{3} - \ldots\ldots \qquad (1)$$

Substituting $-p$ for p in this equation,

$$\log_e (1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \ldots\ldots \qquad (2)$$

Subtracting equation (2) from equation (1), and observing that

$$\log_e (1+p) - \log_e (1-p) = \log_e \left(\frac{1+p}{1-p}\right),$$

we obtain

$$\log_e \left(\frac{1+p}{1-p}\right) = 2\left(p + \frac{p^3}{3} + \frac{p^5}{5} + \ldots\ldots\right) \qquad (3)$$

Let, now, $\frac{1+p}{1-p} = 1 + \frac{1}{z}$, or $p = \frac{1}{2z+1}$; then

$$\log_e \left(1 + \frac{1}{z}\right) = \log_e \left(\frac{z+1}{z}\right) = \log_e (z+1) - \log_e z.$$

Substituting these values in equation (3), we have

$$\log_e(z+1)-\log_e z=2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\ldots.\right) \quad (C)$$

407. This series, (C), may be employed in computing the Napierian logarithm of any number, when the like logarithm of the preceding number is known.

But, by Art. 396, the logarithm of 1 is 0; therefore, making $z+1$ successively equal to the prime numbers 2, 3, 5, 7, etc., and observing that the logarithm of any composite number is equal to the sum of the logarithms of its prime factors (Art. 399), we obtain the following

NAPIERIAN LOGARITHMS.

$$\log_e 2 = 2\left(\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\ldots..\right) = 0.693147$$

$$\log_e 3 = \log_e 2+2\left(\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\frac{1}{7\cdot 5^7}+\ldots\right) = 1.098612$$

$$\log_e 4 = 2\log_e 2 = 1.386294$$

$$\log_e 5 = \log_e 4+2\left(\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\frac{1}{7\cdot 9^7}+\ldots\right) = 1.609438$$

$$\log_e 6 = \log_e 2+\log_e 3 = 1.791759$$

$$\log_e 7 = \log_e 6+2\left(\frac{1}{13}+\frac{1}{3\cdot 13^3}+\frac{1}{5\cdot 13^5}+\ldots.\right) = 1.945910$$

$$\log_e 8 = 3\log_e 2 = 2.079442$$

$$\log_e 9 = 2\log_e 3 = 2.197225$$

$$\log_e 10 = \log_e 2+\log_e 5 = 2.302585$$

etc. etc. etc.

Napierian logarithms are sometimes called *hyperbolic* logarithms, from having been originally derived from the hyperbola. They are also sometimes called *natural* logarithms, from being those which occur first in the investigation of a method of calculating logarithms.

COMMON LOGARITHMS.

408. To compute common logarithms, it is necessary to find the value of M, the modulus of the system, when the base is 10.

Let $$10^x = n, \tag{1}$$

where x is the common logarithm of n.

Then, taking the Napierian logarithm of each member, by Art. 401 we have

$$x \log_e 10 = \log_e n,$$

whence $$x = \frac{\log_e n}{\log_e 10} = \log_e n \times \frac{1}{\log_e 10}, \tag{2}$$

where $\frac{1}{\log_e 10}$ is the factor depending upon the base (Art. 405), and, therefore, the modulus required.

Hence, $$M = \frac{1}{\log_e 10}.$$

But, by Art. 407,

$$\log_e 10 = 2.302585,$$

whence, $$M = \frac{1}{2.302585} = .4342945. \tag{3}$$

409. *The common logarithm of a number may be derived from the Napierian logarithm of that number, by multiplying it by* .4342945.

For, by equation (2), Art. 408, the common logarithm of a number is equal to the Napierian logarithm of that number multiplied by the modulus of the common system. Thus, representing a common logarithm by log,

$$\log 2 = 0.693147 \times .4342945 = 0.301030$$
$$\log 3 = 1.098612 \times .4342945 = 0.477121$$
etc. etc. etc.

410. A table of common logarithms, exact to any number of decimal places, may also be constructed directly from the series already employed. For, multiplying both members of (C) by the modulus of the common system, we have

$$\log(z+1) - \log z = .868589\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \frac{1}{7(2z+1)^7} + \cdots\right) \quad (D)$$

Make $z+1$ successively equal to 2, 3, etc., and we find

$$\log 2 = .868589\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \cdots\right) = 0.301030$$

$$\log 3 = \log 2 + .868589\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \cdots\right) = 0.477121$$

$$\log 4 = 2 \log 2 = 0.602060$$

$$\log 10 = \log 2 + \log 5 = 1.000000$$

etc. etc. etc.

411. *In the common system of logarithms, if the logarithm of any number be known, we can immediately determine the logarithm of the product or quotient of that number by any power of* 10.

For, $\log(m \times 10^n) = \log m + \log 10^n = \log m + n,$

and $\log\left(\frac{m}{10^n}\right) = \log m - \log 10^n = \log m - n.$

That is, if the logarithm of any number be known, we can determine the logarithm of any number which has the same succession of figures, but differs merely by the position of the decimal point (Arts. 394, 395). Thus, if

The logarithm of	15	is	1.176091
"	1.5	"	0.176091
"	.15	"	$\bar{1}$.176091
"	.015	"	$\bar{2}$.176091

Here, the negative sign placed over the characteristic indicates that it alone is negative, the decimal part being always positive.

Common logarithms are often called Briggs's logarithms, from the name of their inventor.

Base of the Napierian System.

412. Let $e^x = 10$,

where e is the Napierian base, and x is the Napierian logarithm of 10. Then, taking the common logarithm of each member, we have

$$x \log e = 1, \text{ or } \log e = \frac{1}{\log_e 10} = .4342945 \quad \text{(Art. 408.)}$$

That is, the modulus of the common system is the common logarithm of the Napierian base. From Arts. 409, 410, it is evident that the base of the Napierian system is between 2 and 3. Calculated to seven places of decimals it is found that

$$e = 2.7182818.$$

TABLES OF COMMON LOGARITHMS.

413. A Table of Logarithms usually contains all the whole numbers between 1 and a given number, with their logarithms.

The characteristics of the first 100 numbers are inserted, but for all other numbers the decimal part only of the logarithms is given, while the characteristic is left to be supplied by inspection (Arts. 394, 395).

The numbers are in the column headed N, and their logarithms, or the decimal parts of their logarithms, are opposite on the same line. Sometimes there is a column headed D, in which are the mean or average differences of the ten logarithms against which they are placed.

TABLE I.

N.	Log.	N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1,799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826076	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851259	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

TABLE II.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775	416
5	021189	021603	022016	022429	022841	023252	023664	024075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021	404
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
1	5323	5714	6105	6495	6885	7276	7664	8053	8442	8830	390
2	9218	9606	9993	050380	050766	051153	051539	051924	052309	052694	386
3	053078	053463	053846	4230	4613	4996	5379	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8905	9185	9563	9942	060320	379
5	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	070039	070407	070776	071145	071514	370
8	071882	072250	072617	072985	073352	9718	4085	4451	4816	5182	366
9	5547	6912	6276	6640	7004	7368	7731	8094	8457	8919	363
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082428	360

The logarithms in the preceding tables are taken in the common system, and are sufficient to illustrate the use of a larger table, such as that appended to the author's Trigonometry. In Table II. the first two figures of several successive numbers, and of the logarithms, when the same, are left to be supplied.

414. Numbers may be formed in Table II. by annexing each figure at the top of the table to the figures in the column at the left, and the decimal of the logarithm of the number formed will be in a line with the figures taken at the left, and in the column with the figure taken at the top. Thus,

The logarithm of	1129	is	3.052694
"	10.95	"	1.039414
"	.1209	"	$\bar{1}$.082426

415. When a number is not in the table, if its factors are there, its logarithm may be found by taking out from the table the logarithms of the factors, and adding them together (Art. 399). Thus, since

$$7248 = 1208 \times 6, \quad \log 7248 = \log 1208 + \log 6$$
$$= 3.082067 + 0.778151 = 3.860218.$$

416. When a number is intermediate between those in the table, its logarithm may be found by interpolation.

For, the tabulated logarithms are of the nature of a regular series, hence they may be interpolated by the formula in Art. 387. Thus,

Let it be required to find the logarithm of 11.887.

Here, 11.887 is a number between 11.88 and 11.89 ; then

$$\log 11.89 - \log 11.88 = 1.075182 - 1.074816 = .000366.$$

Hence, put $d_1 = .000366$, and $p = .7$, and we have

$$\begin{aligned} a &= \log 11.88 &&= 1.074816 \\ p\,d_1 &= .000366 \times .7 &&= .000256 \\ a + p\,d_1 &= \log 11.887 &&= 1.075072 \end{aligned}$$

This process is based upon the supposition that the differences of logarithms are proportional to the differences of their corresponding numbers, which is not strictly correct, yet sufficiently exact for practical purposes.

The difference d_1 might have been taken directly from the table, being found in a line with the logarithm of 11.88, and in column D.

417. When a given logarithm is intermediate between those found in the table, its corresponding number may be found.

For, this is obviously the converse of the last case, and, therefore depends upon a process based upon a like supposition. Thus,

Let it be required to find the number corresponding to the logarithm 1.075072.

The decimal of the given log is .075072

" the log next less " .074816, corresp'ng no. 11.88

Difference, 256.0

Diff. betw. log 11.88 and log 11.89, 366 = 7

Whence, the required number = 11.887

EXAMPLES.

1. Find the logarithm of 1095. Ans. 3.039414.

2. Find the logarithm of .0735. Ans. $\bar{2}.866287$.

Here, $.0735 = \frac{49 \times 15}{10000}$ (Arts. 411, 415).

3. Find the logarithm of 1728. Ans. 3.237543.

4. Find the logarithm of .00016. Ans. $\overline{4}.204120$.

5. Find the logarithm of 2.34. Ans. 0.369216.

6. Find the logarithm of $\frac{37}{94}$. Ans. $\overline{1}.595074$.

Here, by Art. 400,

$$\log \tfrac{37}{94} = \log 37 - \log 94.$$

7. Find the logarithm of $\frac{3}{107}$. Ans. $\overline{2}.447737$.

8. Find the logarithm of $17\frac{14}{23}$. Ans. 1.245727.

Here,

$$17\tfrac{14}{23} = \tfrac{405}{23}; \quad \log \tfrac{405}{23} = \log 81 + \log 5 - \log 23.$$

9. Find the logarithm of 1.0375. Ans. 0.015988.

10. Find the logarithm of 104.857. Ans. 2.020597.

11. Find the number whose logarithm is 0.025306. Ans. 1.06.

12. Find the number whose logarithm is 3.010724.

13. Find the number whose logarithm is 0.009663. Ans. 1.0225.

14. Find the number whose logarithm is 3.009264. Ans. 1021.56.

EXPONENTIAL EQUATIONS.

418. An EXPONENTIAL QUANTITY is one that has an unknown quantity for an exponent; as,

$$a^x, \quad a^{\frac{1}{x}}, \quad x^x, \quad \text{etc.}$$

419. An EXPONENTIAL EQUATION is one containing an exponential quantity, as

$$a^x = b, \quad x^x = a, \quad \text{etc.}$$

Equations of this kind may be readily solved by logarithms.

420. To solve an exponential equation in the form of

$$a^x = b,$$

take the logarithm of each member, and we have

$$x \log a = \log b;$$

whence,

$$x = \frac{\log b}{\log a}.$$

EXAMPLES.

1. Find the value of x in the equation $2^x = 1024$.

Here,

$$x \log 2 = \log 1024;$$

whence,

$$x = \frac{\log 1024}{\log 2} = \frac{3.010300}{0.301030} = 10,$$

which may be verified by involution.

2. Find the value of x in the equation $3^x = 15$.

Ans. 2.465.

3. Find the value of x in the equation $5^x = 100$.

Ans. 2.861.

4. Find the value of x in the equation $a b^x = c$.

Ans. $\frac{\log c - \log a}{\log b}$.

5. Find the value of x in the equation $3^{\frac{4}{x}} = 9$.

Here, raising both members to the xth power, we have

$$3^4 = 9^x, \text{ or } 81 = 9^x; \text{ and } \log 81 = x \log 9;$$

whence,

$$x = 2, \text{ Ans.}$$

6. Find the value of x in the equation $6^{\frac{2}{x}} = \frac{5}{2}$.

Ans. 3.911.

7. Find the value of x in the equation $9^{3^x} = 1000$, there being given the logarithm of $3.14 = 0.496930$.

Here, x is the exponent of the exponent 3; hence, taking the logarithm of each member, we have

$$3^x \log 9 = \log 1000; \quad 3^x = \frac{\log 1000}{\log 9} = 3.14;$$

$$x = \frac{\log 3.14}{\log 3} = 1.04, \text{ Ans.}$$

8. Find the value of x in the equation $\frac{a b^x + c}{d} = n$.

Ans. $\frac{\log (dn - c) - \log a}{\log b}$.

9. From the formulas given in Arts. 340, 341, obtain the value of n in terms of a, r, and l.

Ans. $n = \frac{\log l - \log a}{\log r} + 1$.

10. Given a, r, and S, to find n.

Ans. $n = \frac{\log [a + (r-1) S] - \log a}{\log r}$.

11. Given a, l, and S, to find n.

Ans. $\frac{\log l - \log a}{\log (S-a) - \log (S-l)} + 1$.

12. Given r, l, and S, to find n

Ans. $\frac{\log l - \log [lr - (r-1) S]}{\log r} + 1$.

421. To solve an exponential equation in the form of

$$x^x = a.$$

Taking the logarithms of both members, we have

$$x \log x = \log a,$$

where the value of x may be obtained by trial, as follows:

Find by trial two numbers nearly equal to x, substitute them for x, and note the results. Then,

As the difference of the results is to the difference of the two assumed numbers, so is either error to the correction required in the corresponding assumed number.

EXAMPLES.

1. Required the approximate value of x in the equation $x^x = 100$.

Here, $x \log x = \log 100 = 2$;

and upon trial x is found to be between 3 and 4, but greater than 3.5, and less than 3.6.

Assuming, then, 3.5 and 3.6 for the two numbers, we have,

FIRST SUPPOSITION.		SECOND SUPPOSITION.	
$\log x = \log 3.5 =$	0.544068	$\log x = \log 3.6 =$	0.556303
Multiplying by	3.5	Multiplying by	3.6
$x \log x$	$= 1.904238$	$x \log x$	$= 2.002691$

Diff. results : *Diff. assumed nos.* :: *Error 2d result* : *Its correction.*

.098453 : 0.1 :: .002691 : .00273

Whence, $x = 3.6 - .00273 = 3.59727$.

2. Required the approximate value of x in the equation $x^x = 10$. Ans. 2.506.

3. Find the value of x in the equation $x^x = 256$. Ans. 4.

4. Required the approximate value of x in the equation $x^x = 5$. Ans. 2.129.

COMPOUND INTEREST AND ANNUITIES.

422. Logarithms can be used very advantageously in many of the operations relating to Compound Interest, Annuities, and Increase of Population, as well as Geometrical Progression (Art. 404); but when the unknown quantity occurs as an exponent, they become essential to a direct solution (Art. 420).

423. *To find the amount of a given sum for any time, at compound interest.*

Let P = any principal, in dollars,
r = the interest of one dollar for one year,
and n = the number of years P draws interest.

If A = the amount of P for n years, at the rate r, we have

$$A = P(1+r)^n; \tag{A}$$

whence, $$\log A = \log P + n \log (1+r) \tag{1}$$

$$\log P = \log A - n \log (1+r) \tag{2}$$

$$\log (1+r) = \frac{\log A - \log P}{n} \tag{3}$$

$$n = \frac{\log A - \log P}{\log (1+r)} \tag{4}$$

424. Suppose the interest is due more frequently than *once* a year, for example, quarterly; then $4n$ will be the number of payments and $\frac{r}{4}$ the rate of interest, and the amount of P in n years becomes

$$A = P\left(1+\frac{r}{4}\right)^{4n}.$$

Similarly, if interest be due q times a year, at compound interest, we have

$$A = P\left(1+\frac{r}{q}\right)^{qn}. \tag{B}$$

425. *To find the amount of an annuity, left unpaid for any number of years, at compound interest.*

Let a represent the annuity, and A its amount at the end of n years, at the rate r.

At the end of the first year, a is due; at the end of the second year, $a+a(1+r)$ is due; and so on. Hence, the amount due at the end of n years is

$$a + a(1+r) + a(1+r)^2 + \ldots\ldots + a(1+r)^{n-1},$$

a geometrical series, whose sum gives (Art. 341)

$$A = \frac{a[(1+r)^n - 1]}{r}. \qquad (C)$$

426. *To find the present value of an annuity, to continue for a certain number of years, at compound interest.*

If P represent the present value of an annuity, it must be such a sum as, being at compound interest for n years, at the given rate r, will produce the same amount as the annuity; that is (Arts. 423, 425),

$$P(1+r)^n = \frac{a[(1+r)^n - 1]}{r},$$

whence

$$P = \frac{a[(1+r)^n - 1]}{r(1+r)^n}. \qquad (D)$$

427. Formula (D) may be thus expressed,

$$P = \frac{a}{r} - \frac{a}{r(1+r)^n}.$$

Suppose, now, the annuity to be perpetual, or $n = \infty$, and we have

$$P = \frac{a}{r}. \qquad (E)$$

428. Problems relating to the natural increase of population in a country, being analogous to those in compound interest, are solved by the same formulas.

PROBLEMS.

1. Find the value of n in (C).

Ans. $n = \dfrac{\log(a + rA) - \log a}{\log(1+r)}$.

2. Find the value of n in (D).

Ans. $n = \dfrac{\log a - \log(a - rP)}{\log(1+r)}$.

3. In how many years will any sum of money double itself, at 6 per cent., compound interest?

Ans. $\frac{\log 2 - \log 1}{\log 1.06} = 11.9$, years.

4. Required the amount of $16 for 30 years, at 5 per cent., compound interest. Ans. $69.15.

5. What is the amount of $500 for 9 years, at 6 per cent. per annum, interest semiannually?

Given $2.930036 = \log 851.21$. Ans. $851.21.

6. In what time will $2233.57, at 6 per cent., compound interest, amount to $4000?

Given $\log 2233.57 = 3.349000$. Ans. 10 years.

7. What principal, at 6 per cent., compound interest, will amount to $4000 in 10 years? Ans. $2233.57.

8. At what rate per cent. will $2233.57 amount to $4000, in 10 years, at compound interest?

Ans. 6 per cent.

9. If the population of a certain country increases at the rate of 20 per cent. every 10 years, in what time will it double itself? Ans. 38 years, nearly.

10. What is the present value of an annuity of $500 a year, to be continued 10 years, reckoning at 6 per cent., compound interest? Ans. $3680.04.

Given $0.253060 = \log 1.79085$; $\log .79085 = \bar{1}.898095$; $3.565854 = \log 3680.04$.

11. Required the present value of a perpetual lease, yielding $1200 a year, payable semiannually, interest being at the rate of 5 per cent. Ans. $24000.

12. The present value of an annuity, to be continued 10 years, at 6 per cent., compound interest, payable annually, is $3680.04; required the annuity. Ans. $500.

13. A gentleman dying, left for the benefit of his widow $8000 in an annuity office, paying 5 per cent., compound interest; in how many years will this sum be exhausted, if she draws out, annually, $850?

Ans. 13 years 12 days.

14. Suppose a salary of $600 per year to remain unpaid $5\frac{1}{2}$ years; to what sum will it amount, at 6 per cent. per annum, compound interest, if the salary and interest are payable quarterly?. Ans. $3875.60, nearly.

Given $0.142252 = \log 1.38756$.

GENERAL THEORY OF EQUATIONS.

429. The General Form of a complete equation of the nth degree is

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0,$$

in which n is a positive integer, and the number of terms is $n + 1$. The known quantities $p, q, \ldots\ldots t, u, v$ are either positive or negative, integral or fractional; but the coefficient of x^n is understood to be $+1$, unless otherwise indicated.

The *absolute term*, v, is independent of x, but it may be regarded as the coefficient of x^0 (Art. 77).

430. In reducing an equation to the general form, all the terms must be transposed to the first member, and arranged according to the powers of x. If x^n has a coefficient, it may be removed by dividing the equation by that coefficient; and if any power of x is wanting, it may be supplied, with the coefficient 0. Thus, when reduced to the general form, the equation $4x^2 - 6 = 3x^4$ becomes

$$x^4 \pm 0\,x^3 - \tfrac{4}{3}x^2 \pm 0\,x + 2 = 0.$$

431. A ROOT of an equation is any real or imaginary expression, which, being substituted for its unknown quantity, satisfies the equation, or makes the first member equal to 0 (Art. 148).

Every equation must have at least one root, for, from the nature of an equation, in order to be true, there must be some quantity that will satisfy it, when substituted for its unknown quantity.

432. A FUNCTION of a quantity is any expression which contains that quantity, and is, therefore, dependent upon it. Thus,

$$x^5, \quad x + a, \quad \text{and} \quad 10x^3 - ax^2 + 8,$$

are functions of x.

433. A PERMANENCE of sign occurs when two successive terms of a series have the *same* sign; and

A VARIATION of sign occurs when two successive terms of a series have *different* signs.

GENERAL PROPERTIES.

DIVISIBILITY OF EQUATIONS.

434. *If* a *is a root of an equation in the form*

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0,$$

then the first member is divisible by x — a.

It is evident that the division of the first member by $x - a$ may be carried on until x disappears from the remainder. Let Q represent the quotient, and R the remainder, which must be independent of x; then the given equation may be made to take the form

$$(x - a)\,Q + R = 0.$$

But if $x = a$, then $(x - a)\ Q = 0$, and, consequently,

$$R = 0;$$

that is, $x - a$ is a factor of the first member of the given equation.

435. *If the first member of an equation in the form*

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0$$

is divisible by x — a, *then* a *is a root of the equation.*

For, if the first member of the given equation is divisible by $x - a$, then the equation may be made to take the form

$$(x - a)\ Q = 0.$$

But the substitution of a for x causes the first member of this form to become 0; therefore it will also satisfy the given equation in its original form (Art. 431).

EXAMPLES.

1. Prove that 3 is a root of the equation

$$x^3 - 6\,x^2 + 11\,x - 6 = 0.$$

Here, dividing $x^3 - 6\,x^2 + 11\,x - 6$ by $x - 3$, we obtain $x^2 - 3\,x + 2$, and no remainder.

2. Prove that -1 is a root of the equation $x^3 + 1 = 0$.

Here, $(x^3 + 1) \div (x + 1) = x^2 - x + 1$.

3. Prove by division that 1 is a root of the equation $x^3 + x^2 - 17\,x + 15 = 0$.

4. If -2 is a root of the equation $x^4 - 3x^2 + 4x + 4 = 0$, by what factor must its first member be divisible?

5. What number is a root of the equation

$$(x + 3)\ (x^2 - 2\,x + 5) = 0\ ?$$

NUMBER OF ROOTS.

436. *Every equation of the* n*th degree, containing but one unknown quantity, has* n *roots, and no more.*

Let a be a root of the equation

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0;$$

then, by Art. 434, the first member is divisible by $x - a$, and the equation may take the form

$$(x - a)\,(x^{n-1} + p_1\,x^{n-2} + \ldots\ldots + u_1\,x + v_1) = 0.$$

This equation may be satisfied by making either factor of the first member equal to 0; hence,

$$x^{n-1} + p_1\,x^{n-2} + \ldots\ldots + u_1\,x + v_1 = 0.$$

But this equation must have some root, as b, and may be placed under the form

$$(x - b)\,(x^{n-2} + p_2\,x^{n-3} + \ldots\ldots + u_2\,x + v_2) = 0;$$

and be satisfied by making the second factor equal to 0; and so on.

Since each of the factors, $x - a$, $x - b$, etc., contains only the first power of x, it is evident that the original equation can be separated into as many such binomial factors as there are units in the exponent of the highest power of the unknown quantity, and no more; that is, into n factors, or

$$(x - a)\,(x - b)\,(x - c)\,\ldots\ldots\,(x - l) = 0.$$

Hence, by Art. 435, the given equation has the n roots $a, b, c, \ldots\ldots l$.

Moreover, if the equation had another root, as r, then it must contain another factor, $x - r$; which is impossible.

437. It should be observed, that the n binomial factors of which the general equation of the nth degree is composed are not necessarily *unequal;* hence, two or more of the roots of an equation may be equal. Thus, the equation

$$x^3 - 6x^2 + 12x - 8 = 0$$

may be factored so as to take the form

$$(x-2)(x-2)(x-2) = 0, \quad \text{or} \quad (x-2)^3 = 0;$$

and, hence, the three roots are 2, 2, and 2.

438. An equation is satisfied by making *any one* of the factors of its first member equal to 0; but two or more factors cannot each be made equal to 0 at the same time, and for the same value of x, unless there are equal roots.

439. It will be seen that any equation, one of whose roots is known, may be readily depressed to another of the next lower degree, which shall contain the remaining roots. Hence, if all the roots of an equation are known except two, those may be obtained from the depressed equation, by the rules for quadratics.

Examples.

1. One root of the equation $x^3 + 2x^2 - 23x - 60 = 0$ is -3; required the depressed equation, and the remaining roots.

Here, dividing $x^3 + 2x^2 - 23x - 60$ by $x + 3$, we obtain $x^2 - x - 20$; hence $x^2 - x - 20 = 0$ is the depressed equation. Solving this equation by the rules for quadratics, we obtain 5 and -4 as the remaining roots.

Note. If we were to divide the original equation by $x + 3 = 0$, we should obtain $x^2 - x - 20 = \frac{0}{0}$; that is, when $x + 3 = 0$, the original equation is satisfied, whatever the value of the other factor, $x^2 - x - 20$. When $x + 3 = 0$, $x^2 - x - 20$, or $\frac{0}{0}$, must be equal to -8, for the same value of x. But the original equation is satisfied by making either of its factors equal to 0; hence, we may take $x^2 - x - 20 = 0$, instead of $x + 3 = 0$ (Art. 438).

2. One root of the equation $x^3 - 19x + 30 = 0$ is 2; required the depressed equation, and the remaining roots.

Ans. Equation, $x^2 + 2x - 15 = 0$; Roots, 3 and -5.

3. Required the three roots of the equation $x^3 = a^3$, or $x^3 - a^3 = 0$.

Ans. a and $\frac{a}{2}(-1 \pm \sqrt{-3})$.

4. One root of the equation $x^3 + x^2 - 16x + 20 = 0$ is -5; required the remaining roots. Ans. 2 and 2.

5. Two roots of the equation $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0$ are 1 and 2; required the remaining roots.

Ans. 4 and -4.

6. One root of the equation $x^4 - 7x^3 + 3x + 3 = 0$ is 1; what equation contains the remaining roots?

FORMATION OF EQUATIONS.

440. *An equation having any given roots may be formed by subtracting each root from the unknown quantity, and placing the product of these binomial factors equal to* 0.

For, it is evident, from principles already established, that an equation having the n roots $a, b, c, \ldots\ldots l$ may take the form

$$(x - a)(x - b)(x - c) \ldots\ldots (x - l) = 0.$$

After performing the multiplication indicated, the equation will assume the form

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0.$$

Examples.

1. Form the equation whose roots are 1, 2, and -4.

Here, $(x - 1)(x - 2)(x + 4) = x^3 + x^2 - 10x + 8$; hence, the required equation is $x^3 + x^2 - 10x + 8 = 0$.

2. What equation has the roots -1, -3, and -5?
Ans. $x^3 + 9x^2 + 23x + 15 = 0$.

3. What equation has the roots 1, 2, 3, and 4?
Ans. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

4. Form the equation whose roots are 1, $\frac{1}{2}$, and $\frac{1}{3}$.
Ans. $x^3 - \frac{11}{6}x^2 + x - \frac{1}{6} = 0$.

5. Form the equation whose roots are 4, 4, and 5.
Ans. $x^3 - 13x^2 + 56x - 80 = 0$.

6. Form the equation whose roots are 5, -2, and -3.
Ans. $x^3 - 19x - 30 = 0$.

7. Form the equation whose roots are ± 1 and ± 2.
Ans. $x^4 - 5x^2 + 4 = 0$.

8. Form the equation whose roots are 0, -1, 3, and 4.
Ans. $x^4 - 6x^3 + 5x^2 + 12x = 0$.

COMPOSITION OF COEFFICIENTS.

441. *The coefficient of the second term of an equation is the sum of all the roots, with their signs changed; that of the third term is the sum of their products, taken two and two; that of the fourth term is the sum of their products, taken three and three, with their signs changed, etc.; and the last term is the product of all the roots, with their signs changed.*

For, resuming the equation

$$(x-a)\,(x-b)\,(x-c)\,\ldots\ldots\,(x-k)\,(x-l) = 0,$$

if we perform the multiplication indicated, we obtain

$$(x-a)\,(x-b) = x^2 - (a+b)\,x + a\,b,$$

$$(x-a)\,(x-b)\,(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)\,x - abc,$$

and so on. When n factors have been multiplied, the coefficients of the general equation become

$$p = -a - b - c - \ldots\ldots - k - l$$
$$q = ab + ac + bc + \ldots\ldots + kl$$
$$r = -abc - abd - acd - \ldots\ldots - ikl$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$v = \pm abc \ldots\ldots kl$$

which correspond with the enunciation of the proposition.

442. If $p = 0$, that is, if the second term of an equation be wanting, the sum of the roots will be 0.

If $v = 0$, that is, if the absolute term of an equation be wanting, at least one root must be 0.

443. Every rational root of an equation is a divisor of the last term.

444. When all the roots of an equation but two are known, the coefficient of the second term of the depressed equation (Art. 439) can be found by subtracting the sum of the known roots, with their signs changed, from the coefficient of the second term of the original equation. The absolute term can be found by dividing the absolute term of the original equation by the product of the known roots, with their signs changed.

Examples.

1. What is the sum of the roots of the equation $x^3 - 7x + 6 = 0$?

2. Are 2, 3, and 4 the roots of the equation $x^3 - 8x^2 + 26x - 24 = 0$? (Art. 441.) If not, how must the equation be changed in order that they shall be its roots?

3. What are the second and last terms of the equation whose roots are 3, 5, -2, and $\frac{2}{3}$?

Ans. $-\frac{29}{3}x^3$ and -20.

4. What are the third and last terms of the equation whose roots are -2, -3, and 4?

Ans. $-14x$ and -24.

5. Write the equation whose roots are 1, 2, and 3.

Ans. $x^3 - 6x^2 + 11x - 6 = 0$.

6. What are the roots of the equation $x^4 + 8x^3 + 10x^2 = 0$?

Ans. 0, 0, and $-4 \pm \sqrt{6}$.

7. Two roots of the equation $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0$ are 1 and 2; obtain the depressed equation by Art. 444.

FRACTIONAL ROOTS.

445. *An equation whose coefficients are all integers, that of the first term being unity, cannot have a root equal to a rational fraction.*

If possible, let $\frac{a}{b}$, a rational fraction in its lowest terms, be a root of the equation

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0,$$

in which $p, q, \ldots\ldots t, u, v$ are integers.

Then $\left(\frac{a}{b}\right)^n + p\left(\frac{a}{b}\right)^{n-1} + q\left(\frac{a}{b}\right)^{n-2} + \ldots\ldots + t\left(\frac{a}{b}\right)^2 + u\left(\frac{a}{b}\right) + v = 0.$

Multiplying the equation by b^{n-1}, and transposing,

$$\frac{a^n}{b} = -(pa^{n-1} + qa^{n-2}b + \ldots\ldots + ta^2b^{n-3} + uab^{n-2} + vb^{n-1}).$$

Now, since a and b have no common factor, a^n and b can have none; hence, we have a fraction in its lowest terms equal to an entire quantity, which is impossible. Therefore, no root of the equation can be a rational fraction.

446. An equation whose coefficients are whole numbers, although its *rational* roots are likewise whole numbers, may have *real* roots whose value may be expressed in the form of irrational fractions, or *approximately* expressed by means of decimal fractions.

A real root which cannot be exactly expressed in numbers, is said to be *incommensurable*.

IMAGINARY ROOTS.

447. *If the coefficients of an equation be real quantities, imaginary roots enter it by pairs.*

Suppose $a + b\sqrt{-1}$ to be a root of the equation $x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0$.

If we substitute this root in place of x, and develop each expression by the binomial theorem, we find that all the *odd* terms of each series either contain the *even* powers of $b\sqrt{-1}$, or simply the powers of a, and are therefore real, while all the *even* terms contain the *odd* powers of $b\sqrt{-1}$, and are therefore imaginary. Representing the sum of all the real quantities by P, and that of the imaginary quantities by $Q\sqrt{-1}$, we have

$$P + Q\sqrt{-1} = 0.$$

This equation can be true only when both $P = 0$, and $Q = 0$.

If we now substitute $a - b\sqrt{-1}$ for x, we find that the series differ from the former only in having their *even* terms *negative*. Hence, we obtain

$$P - Q\sqrt{-1} = 0,$$

which must be a true equation, for we have already shown that P and Q are both 0. Therefore, if $a + b\sqrt{-1}$ is a root of any equation, $a - b\sqrt{-1}$ is also a root of the same equation.

Again, if $+ b\sqrt{-1}$ is a root of any equation, then will $- b\sqrt{-1}$ be a root of the same equation. For, if $b\sqrt{-1}$ be substituted for x, all the terms of the equation in which x has an even exponent are real, while those in which it has an odd exponent are imaginary; and the substitution of $- b\sqrt{-1}$ produces results differing only in the signs of the imaginary quantities, as above.

448. If the coefficients of an equation be rational quantities, irrational roots *of the form* $a \pm \sqrt{b}$, or $\pm \sqrt{b}$, enter it by pairs.

This may be proved in the same manner as the preceding proposition, substituting *rational* and *irrational* for *real* and *imaginary*.

449. The product of a pair of imaginary quantities is always positive. Thus,

$$(\pm a + b\sqrt{-1})(\pm a - b\sqrt{-1}) = a^2 + b^2,$$

and

$$(+ b\sqrt{-1})(- b\sqrt{-1}) = b^2.$$

450. An equation whose coefficients are real, and whose degree is *odd*, must always have an *odd* number of real roots; and it must therefore always have at least one real root, affected with a sign contrary to that of its last term (Art. 441).

451. If the degree is *even*, the equation must either have an *even* number of real roots, or *none*; but if its last term is negative, it must have at least two real roots, with different signs (Arts. 441, 449).

EXAMPLES.

1. One root of the equation $x^3 + x^2 + 4 = 0$ is -2; what are the others?

Ans. $\frac{1 \pm \sqrt{-7}}{2}$.

2. One root of the equation $x^3 - 12x^2 + 18x + 56 = 0$ is $4 + \sqrt{30}$; what are the others?

Ans. $4 - \sqrt{30}$ and 4.

3. One root of the equation $x^4 - 5x^3 - 2x^2 + 12x + 8 = 0$ is $2 - 2\sqrt{2}$; what are the others?

Ans. $2 + 2\sqrt{2}$, -1, and 2.

4. One root of the equation $x^3 + 7x^2 + 20x + 50 = 0$ is $-1 + 3\sqrt{-1}$; what are the others?

DESCARTES' RULE.

452. *A complete equation cannot have a greater number of positive roots than it has variations of sign; nor a greater number of negative roots than it has permanences of sign.*

Let any complete equation have the following signs,

$$+ \quad + \quad + \quad - \quad + \quad - \quad + \quad - \quad -$$

in which there are three permanences and five variations (Art. 433).

If we introduce a new positive root, we multiply by $x - a$, and the signs of the partial and final products may be expressed thus,

$$\begin{array}{cccccccccc}
+ & + & + & - & + & - & + & - & - & \\
 & - & - & - & + & - & + & - & + & + \\
\hline
+ & \pm & \pm & - & + & - & + & - & \pm & + \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}$$

However the ambiguous sign is interpreted, there must be at least one variation between 1 and 4, and one between 8 and 10; and since the four variations between 4 and 8 remain as in the original, there are at least six variations in the product. Consequently, the introduction of a positive root has caused the introduction of at least one additional variation; and as this is true of any positive root, there must be at least as many variations of sign as there are positive roots.

In the same way, by introducing the factor $x + a$, it may be shown that there must be at least as many permanences of sign as there are negative roots.

453. If an equation is incomplete, the missing terms should be supplied, with the coefficient ± 0 before applying the foregoing rule (Art. 430).

In such an equation, imaginary roots may sometimes be discovered by means of the double sign of 0. Thus, in the equation $x^3 + x^2 \pm 0\,x + 4 = 0$, if we take the upper sign, there is no variation, and no positive root; and if we take the lower sign, there is only one permanence, and but one negative root. Therefore there are two imaginary roots.

In general, whenever the term which precedes the missing one has the same sign as that which follows it, the equation must have imaginary roots. If those terms have different signs, and only a single term is wanting, the equation may, or may not, have imaginary roots, but Descartes' Rule does not detect them. If two or more successive terms of an equation be wanting, the equation must have imaginary roots.

454. In any complete equation, the sum of the number of variations and of permanences is equal to the number of terms less one, or to the degree of the equation (Art. 429). Hence, when the roots are all real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of permanences (Art. 436).

455. A complete equation whose terms are all positive can have no positive root; and one whose terms are alternately positive and negative can have no negative root.

EXAMPLES.

1. What is the limit to the number of positive and of negative roots in the equation $x^3 - 11\,x^2 + 43\,x - 65 = 0$?

Ans. It can have no negative root, and not more than 3 positive ones.

2. The roots of the equation $x^3 \pm 0\,x^2 - 3\,x - 2 = 0$ are all real; what signs must they take?

Ans. One positive; two negative.

3. The roots of the equation $x^3 - 10x + 3 = 0$ are all real; what signs must they take?

Ans. Two positive; one negative.

4. The roots of the equation $x^3 - 7x^2 + 36 = 0$ are all real; what signs must they take?

5. What are the signs of the roots of the equation $x^3 - x^2 - 4 = 0$? Ans. One positive; two imaginary.

6. If the roots of the equation $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0$ are all real, what signs must they take?

Ans. Four positive; one negative.

7. What are the signs of the roots of the equation $x^2 + 16x + 55 = 0$?

TRANSFORMATION OF EQUATIONS.

456. The TRANSFORMATION of an equation, as the term is here used (Art. 149), consists in changing the equation into another of the same degree, whose roots shall bear a specified relation to those of the given equation.

457. *To transform an equation into another which shall have the same roots with their signs changed*

Let the given equation be

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0,$$

Put $x = -y$; then, whatever value x may have, y will have the same, with its sign changed. Substituting this value of x, we have

$$(-y)^n + p(-y)^{n-1} + q(-y)^{n-2} + \ldots\ldots + ty^2 - uy + v = 0,$$

in which the terms having an odd exponent are negative, and the others positive. If n is even, the equation is in its proper form; but if n is odd, all the signs should be changed; in either case, then, we have

$$y^n - p y^{n-1} + q y^{n-2} - \ldots\ldots \pm (+ t y^2 - u y + v) = 0.$$

Hence, to effect the required transformation, we may simply

Change the signs of the alternate terms.

EXAMPLES.

1. The roots of the equation $x^3 - 7x + 6 = 0$ are 1, 2, and -3; find the equation whose roots are -1, -2, and 3.

Here, as the *complete* equation is $x^3 \pm 0x^2 - 7x + 6 = 0$, we must change the signs of $0x^2$ and 6; hence the required equation is $x^3 - 7x - 6 = 0$.

2. Two of the roots of the equation $x^4 - 2x^3 + x - 132 = 0$ are 4 and -3; find the corresponding equation whose roots are 3 and -4.

3. If the roots of the equation $x^3 - 6x^2 + 3x + 10 = 0$ are 5, 2, and -1, what are the roots of the equation $x^3 + 6x^2 + 3x - 10 = 0$?

458. *To transform an equation into another whose roots shall be some multiple of those of the given equation.*

Let the given equation be

$$x^n + p x^{n-1} + q x^{n-2} + \ldots\ldots + t x^2 + u x + v = 0.$$

Put $y = mx$, or $x = \frac{y}{m}$, and we have

$$\left(\frac{y}{m}\right)^n + p\left(\frac{y}{m}\right)^{n-1} + q\left(\frac{y}{m}\right)^{n-2} + \ldots\ldots + t\left(\frac{y}{m}\right)^2 + u\left(\frac{y}{m}\right) + v = 0.$$

Multiplying by m^n,

$$y^n + pmy^{n-1} + qm^2y^{n-2} + \ldots\ldots + tm^{n-2}y^2 + um^{n-1}y + vm^n = 0.$$

Hence, to effect the required transformation, we may

Multiply the coefficient of the second term by the given factor, that of the third term by its square, that of the fourth term by its third power, and so on, for the coefficients of the transformed equation.

459. In a similar manner we may transform an equation into another whose roots shall be equal to those of the given equation divided by some quantity.

EXAMPLES.

1. Transform the equation $x^3 - 2x^2 - x + 2 = 0$ into another whose roots shall be twice as great.

Here, if we substitute $\frac{y}{2}$ for x, we obtain $\frac{y^3}{8} - \frac{y^2}{2} - \frac{y}{2} + 2 = 0$, which, multiplied by 8, becomes $y^3 - 4y^2 - 4y + 16 = 0$. The coefficients of the transformed equation may also be obtained by multiplying the coefficient of x^2 by 2, that of x by 2^2, and the absolute term by 2^3.

2. Transform the equation $x^2 - 7x + 6 = 0$ into another whose roots shall be 3 times as great.

Ans. $y^2 - 21y + 54 = 0$.

3. Transform the equation $x^3 - 7x - 6 = 0$ into another whose roots shall be 4 times as great.

Ans. $y^3 - 112y - 384 = 0$.

4. Transform the equation $x^2 - \frac{3}{2}x - 27 = 0$ into another whose roots shall be one third as great.

Ans. $y^2 - \frac{1}{2}y - 3 = 0$.

460. *To transform an equation containing fractional coefficients into another whose coefficients shall be entire, that of the first term being unity.*

If we assume m (Art. 458) equal to the least common multiple of the denominators found in the equation, it will always remove them; but often a less number will produce the same result. Hence we

Assume such a factor that its proper power (Art. 458) *will cancel the denominator found in any term.*

EXAMPLES.

Transform the following equations into others whose coefficients shall be entire.

1. $x^3 - \frac{1}{2}x^3 - \frac{1}{36}x + \frac{1}{108} = 0.$

Here, the denominators, when factored, are 3, $3^2 . 2^2$, $3^3 . 2^2$; hence the multiplier 3 . 2, or 6, will remove them. Multiplying the four terms, respectively, by 1, 6, 6^2, and 6^3, we obtain $y^3 - 2y^2 - y + 2 = 0$, whose roots are 6 times as great as those of the given equation.

2. $x^3 + \frac{3}{4}x - \frac{7}{4} = 0.$ Ans. $y^3 + 3y - 14 = 0.$

3. $x^4 - 5x^3 + \frac{35}{4}x^2 - \frac{25}{4}x + \frac{3}{2} = 0.$

Ans. $y^4 - 10y^3 + 35y^2 - 50y + 24 = 0.$

4. $x^2 - \frac{m}{n}x + \frac{a}{b} = 0.$ Ans. $y^2 - bmy + ab n^2 = 0.$

5. $x^3 + \frac{1}{30}x^3 - \frac{1}{5}x - \frac{1}{30} = 0.$

Ans. $y^3 + y^2 - 180y - 900 = 0.$

461. *To transform an equation into another whose roots shall be the reciprocals of those of the given equation.*

Let the given equation be

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0.$$

Put $y = \frac{1}{x}$, or $x = \frac{1}{y}$, and we have

$$\frac{1}{y^n} + \frac{p}{y^{n-1}} + \frac{q}{y^{n-2}} + \ldots\ldots + \frac{t}{y^2} + \frac{u}{y} + v = 0.$$

Multiplying by y^n, and reversing the order of the terms,

$$vy^n + uy^{n-1} + ty^{n-2} + \ldots\ldots + qy^2 + py + 1 = 0$$

Dividing by v,

$$y^n + \frac{u}{v}y^{n-1} + \frac{t}{v}y^{n-2} + \ldots\ldots + \frac{q}{v}y^2 + \frac{p}{v}y + \frac{1}{v} = 0.$$

We may, therefore,

Write the coefficients in an inverse order, and divide by the absolute term of the given equation.

Examples.

Transform the following equations into others whose roots shall be the reciprocals of those of the given equations. 0 should be used as a coefficient for terms that are wanting.

1. $x^3-6x^2+11x-6=0$. Ans. $y^3-\frac{11}{6}y^2+y-\frac{1}{6}=0$.
2. $x^4-x^3-3x^2+x+2=0$. Ans. $y^4+\frac{1}{2}y^3-\frac{3}{2}y^2-\frac{1}{2}y+\frac{1}{2}=0$.
3. $x^3-9x^2+\frac{6}{7}x-\frac{1}{49}=0$. Ans. $y^3-42y^2+441y-49=0$.
4. $x^3-4x^2+9=0$. Ans. $y^3-\frac{4}{9}y+\frac{1}{9}=0$.

462. *To transform an equation into another whose roots shall differ from those of the given equation by a given quantity.*

Let the given equation be

$$x^n+px^{n-1}+qx^{n-2}+\ldots\ldots+tx^2+ux+v=0. \quad (1)$$

Put $y=x-r$, or $x=y+r$, and we have

$$(y+r)^n+p(y+r)^{n-1}+\ldots\ldots+u(y+r)+v=0. \quad (2)$$

Developing these expressions by the binomial theorem, we obtain an equation of the form

$$y^n+p'y^{n-1}+q'y^{n-2}+\ldots\ldots+t'y^2+u'y+v'=0, \quad (3)$$

in which

$$p'=nr+p$$

$$q'=n\frac{n-1}{2}r^2+p(n-1)r+q$$

. . . . . .

$$t'=n\frac{n-1}{2}r^{n-2}+p(n-1)\frac{n-2}{2}r^{n-3}+q(n-2)\frac{n-3}{2}r^{n-4}+\ldots\ldots+t$$

$$u'=nr^{n-1}+p(n-1)r^{n-2}+q(n-2)r^{n-3}+\ldots\ldots+2tr+u$$

$$v'=r^n+pr^{n-1}+qr^{n-2}+\ldots\ldots+tr^2+ur+v.$$

The roots of (3) are evidently *less* by r than those of (1). If $-r$ be substituted for r, the roots of the transformed equation will be *greater* by r than those of (1).

When n is quite small, the substitution may be effected as above ; but for equations of a higher degree, a less tedious method is desirable.

463. If we restore x, by substituting $x - r$ for y in (3), we obtain

$$(x-r)^n + p'(x-r)^{n-1} + \ldots\ldots + u'(x-r) + v' = 0, \quad (4)$$

which is evidently identical with (1), and must reduce to the same form when developed ; hence, whatever is proved of (4), is also true of (1). If we divide (4) by $x-r$, we obtain $(x-r)^{n-1} + p'(x-r)^{n-2} + \ldots\ldots + u'$, and a remainder of v' ; if we divide the last quotient by $x-r$, we obtain a remainder of u' ; and so on, until we obtain all the coefficients of (3) as remainders. Hence, to effect the required transformation,

Divide the first member of the given equation by its unknown quantity MINUS *the proposed difference, if the roots of the transformed equation are to be* LESS *than those of the given equation, (or by its unknown quantity* PLUS *the difference if they are to be greater), and the remainder will be the absolute term of the transformed equation.*

Divide the quotient just found by the same divisor, and the remainder will be the coefficient of the first power of the new unknown quantity, and so on, until all the coefficients are thus obtained, in an inverse order.

These successive divisions may be written out in full, but the work can be greatly abbreviated by the use of

SYNTHETIC DIVISION.

464. This method of dividing, which was first suggested by Horner, is an abridgment of the method by detached coefficients (Art. 81). Although synthetic di-

vision may be employed when the divisor has more than two terms, yet we shall have occasion to apply it only in cases where the divisor is a binomial, the coefficient of whose first term is unity.

Let it be required to divide $x^3 - 19x + 30$ by $x - 2$.

OPERATION BY DETACHED COEFFICIENTS.

$$\begin{array}{r|l} 1 \pm 0 - 19 + 30 & 1 - 2 \\ \hline 1 - 2 & 1 + 2 - 15 \\ \hline + 2 - 19 & \\ + 2 - 4 & \\ \hline - 15 + 30 & \\ - 15 + 30 & \\ \hline 0 & \end{array}$$

$x^2 + 2x - 15 =$ Quotient.

It will be observed that the first term of each partial product, and the last term of each partial dividend, may be omitted ; for they are merely repetitions of the figures standing above them. Also, the remaining term of each partial product may be *added* to the corresponding term of the dividend, provided we change the sign of the second term of the divisor before multiplying. The work now stands thus :

$$\begin{array}{r|l} 1 \pm 0 - 19 + 30 & 1 + 2 \\ \hline + 2 & 1 + 2 - 15 \\ \hline + 2 & \\ + 4 & \\ \hline - 15 & \\ - 30 & \\ \hline 0 & \end{array}$$

As the first term of the divisor is 1, it is usually omitted, and the first terms of the dividends constitute the

quotient. Raising, now, the oblique columns, we have the following concise form for the

OPERATION BY SYNTHETIC DIVISION.

Dividend,	$1 \pm 0 - 19 + 30$	$\lfloor + 2$
Partial Products,	$+ 2 + 4 - 30$	
Quotient,	$1 + 2 - 15, + 0$	Remainder.

Here, we use only the second term of the divisor, *with its sign changed;* each term of the quotient is the sum of the vertical column under which it stands, and each term of the second line is obtained by multiplying the next preceding term of the quotient by the divisor, as written.

Synthetic division is especially convenient when there are several successive operations in which each quotient becomes in turn a dividend.

465. The transformation of equations into others whose roots shall differ from those of the given equation by a given quantity, may now be effected by the process of synthetic division, or otherwise.

EXAMPLES.

1. Transform the equation $x^3 + x^2 - 10x + 8 = 0$ into another whose roots shall be less by 4.

FIRST OPERATION.

$$\begin{array}{rrrrr}
x^3 = & (y+4)^3 = & y^3 & + 12y^2 & + 48y + 64 \\
x^2 = & (y+4)^2 = & & y^2 & + 8y + 16 \\
-10x = & -10(y+4) = & & & -10y - 40 \\
8 = & & & & 8 \\
\hline
 & & y^3 & + 13y^2 & + 46y + 48 = 0
\end{array}$$

SECOND OPERATION.

$x^3 + x^2 - 10x + 8 \mid x - 4$

$x^3 - 4x^2 \mid x^2 + 5x + 10 \mid x - 4$

$5x^2 - 10x \mid x^2 - 4x \mid x + 9 \mid x - 4$

$5x^2 - 20x \mid 9x + 10 \mid x - 4 \mid 1$

$10x + 8 \mid 9x - 36 \mid 13$, 3d rem.

$10x - 40 \mid 46$, 2d remainder.

48, 1st remainder.

THIRD OPERATION.

$1 + 1 - 10 + 8 \mid +4$

$+4 + 20 + 40$

1st quotient, $1 + 5 + 10, +48$, 1st remainder, or v'.

$+4 + 36$

2d quotient, $1 + 9, +46$, 2d remainder, or u'.

$+4$

3d quotient, $1, +13$, 3d remainder, or t'.

Hence, $y^3 + 13y^2 + 46y + 48 = 0$ is the required equation.

2. Transform the equation $x^4 - x^2 - 2x - 1 = 0$ into another whose roots shall be greater by 2.

OPERATION.

$1 \pm 0 \quad -1 \quad -2 \quad -1 \mid -2$

$-2 \quad +4 \quad -6 \quad +16$

$-2 \quad +3 \quad -8 \quad +15$, 1st remainder, or v'.

$-2 \quad +8 \quad -22$

$-4 \quad +11 \quad -30$, 2d remainder, or u'.

$-2 \quad +12$

$-6 \quad +23$, 3d remainder, or t'.

-2

-8, 4th remainder, or s'.

Therefore, $y^4 - 8y^3 + 23y^2 - 30y + 15 = 0$ is the required equation.

In the foregoing solution the coefficient 1 is not repeated, and the columns of figures are separated; in other respects it is like the preceding one.

This method may also be applied when the coefficient of the first term of the given equation is not unity.

3. Increase the roots of the equation $6x^3 - 11x^2 + 6x - 1 = 0$ by 0.5.

OPERATION.

6	-11	$+6$	-1	-0.5
	-3	$+7$	-6.5	
6	-14	$+13$	$, -7.5 = v'$	
	-3	$+8.5$		
6	-17	$, +21.5 = u'$		
	-3			
6	$, -20 = t'.$			

Therefore, $6y^3 - 20y^2 + 21.5y - 7.5 = 0$ is the required equation.

4. Transform the equation $x^3 - 9x^2 + 26x - 24 = 0$ into another whose roots shall be less by 1.

Ans. $y^3 - 6y^2 + 11y - 6 = 0$.

5. Diminish the roots of the equation $x^3 - 13x^2 + 56x - 80 = 0$ by 3. Ans. $y^3 - 4y^2 + 5y - 2 = 0$.

6. Increase the roots of $x^4 - 5x^2 + 4 = 0$ by 2.

Ans. $y^4 - 8y^3 + 19y^2 - 12y = 0$.

466. *To transform a complete equation into another whose second term shall be wanting.*

To make $p' = 0$, it is only necessary to make $nr + p = 0$ (Art. 462), or $r = -\frac{p}{n}$. We must, then, substitute $y - \frac{p}{n}$ for x; that is,

Substitute for the unknown quantity a binomial consisting of a new unknown quantity, and the coefficient of the second term, taken with a contrary sign, and divided by the number denoting the degree of the equation.

This substitution may be effected in either of the ways mentioned in Arts. 462–465. It is evident that the sum of the negative roots of the transformed equation must be equal to the sum of the positive roots (Art. 442).

Examples.

Transform the following equations into others whose second terms shall be wanting.

1. $x^3 - 6x^2 + 9x - 6 = 0$.

Assume $x = y + \frac{6}{3} = y + 2$.

FIRST OPERATION.

$$\begin{aligned} x^3 &= y^3 + 6y^2 + 12y + 8 \\ -6x^2 &= \quad -6y^2 - 24y - 24 \\ +9x &= \qquad\quad 9y + 18 \\ -6 &= \qquad\qquad -6 \\ \hline & y^3 \pm 0y^2 - 3y - 4 = 0 \end{aligned}$$

SECOND OPERATION.

$$\begin{array}{rrrrl} 1 & -6 & +9 & -6 & \mid +2 \\ & +2 & -8 & +2 & \\ \hline & -4 & +1 & , -4, & \text{1st remainder.} \\ & +2 & -4 & & \\ \hline & -2 & , -3, & & \text{2d remainder.} \\ & +2 & & & \\ \hline & 0, & & & \text{3d remainder.} \end{array}$$

Hence, the transformed equation is $y^3 - 3y - 4 = 0$, whose roots are *less* by 2.

2. $x^2 - px + q = 0$. Ans. $y^2 - \frac{1}{4}p^2 + q = 0$.

3. $x^3 + 6x^2 - 3x + 4 = 0$. Ans. $y^3 - 15y + 26 = 0$.

4. $x^4 - 8x^3 + 19x^2 - 12x = 0$. Ans. $y^4 - 5y^2 + 4 = 0$.

5. $x^4 - 8x^3 + 23x^2 - 30x + 15 = 0$.
Ans. $y^4 - y^2 - 2y - 1 = 0$.

6. $x^3 + x^2 + 4 = 0$. Ans. $y^3 - \frac{1}{3}y + \frac{1}{27}\frac{1}{0} = 0$.

DERIVED POLYNOMIALS.

467. A DERIVED POLYNOMIAL is one that may be obtained from any given polynomial, containing only a single unknown quantity, as x, by multiplying each term by the exponent of x in that term, and then diminishing the exponent by 1.

The *second derived polynomial* is the derived polynomial of the first derived polynomial; the third is derived in like manner from the second, and so on, each being one degree lower than the preceding.

Derived polynomials are also called *derivatives.*

468. A DERIVED EQUATION is one whose first member is a derivative of the first member of another equation. The first derived equation is also called the *limiting* or *separating equation*, when used to separate, or determine the limits of the roots of an equation.

EXAMPLES.

1. Obtain the successive derived polynomials of $x^3 + 5x^2 + 3x + 9$.

Ans. { First, $3x^2 + 10x + 3$.
Second, $6x + 10$.
Third, 6.
Fourth, 0.

As 9 may be regarded as $9x^0$, we have 0×9 for a

coefficient, which causes the known term to disappear in the derivative. The term $3x$ becomes $1 \times 3x^0$, or 3.

2. What are the derivatives of $2x^4 - 3x^3 - 4x^2 + 5x - 6$?

3. Obtain the first derived equation of $x^3 - 5x^2 + 6x - 2 = 0$. Ans. $3x^2 - 10x + 6 = 0$.

469. If we use R for v', R_1 for u', $\frac{R_2}{2}$ for t', etc., equation (3) (Art. 462), by reversing the order of the terms, becomes

$$R + R_1 y + \frac{R_2}{2} y^2 + \frac{R_3}{2 \cdot 3} y^3 + \ldots\ldots + y^n = 0, \qquad (5)$$

in which

$$R = r^n + p r^{n-1} + \ldots\ldots + t r^2 + u r + v$$
$$R_1 = n r^{n-1} + p (n-1) r^{n-2} + \ldots\ldots + 2 t r + u$$
$$R_2 = n (n-1) r^{n-2} + p (n-1) (n-2) r^{n-3} + \ldots\ldots + 2 t$$
$$R_3 = n (n-1) (n-2) r^{n-3} + p (n-1) (n-2) (n-3) r^{n-4} + \ldots\ldots$$

It will be seen that R, the absolute term, or coefficient of y^0, takes the same *form* as the original equation, (1) (Art. 462), r being substituted for x; R_1 takes the *form* of the first derivative of (1), R_2 that of its second derivative, and so on. Hence the coefficients of the transformed equations of Arts. 462–466 may be obtained by means of the law of derived polynomials.

Examples.

1. Transform the equation $x^3 - 7x - 6 = 0$ into another whose roots shall be greater by 3.

Here, $r = -3$. The successive derivatives of the first member are $3x^2 - 7$, $6x$, and 6. Hence,

$$R = r^3 - 7r - 6 = -27 + 21 - 6 = -12$$
$$R_1 = 3r^2 - 7 = 27 - 7 = 20$$
$$R_2 = 6r = -18; \quad R_3 = 6.$$

Substituting these values in (5), we obtain

$$-12 + 20y - \tfrac{18}{2} y^2 + \tfrac{6}{6} y^3 = 0, \text{ or } y^3 - 9y^2 + 20y - 12 = 0.$$

2. Transform the equation $x^4 - 8x^3 + 24x^2 - 32x - 65 = 0$ into another whose roots shall be less by 4.

Ans. $y^4 + 8y^3 + 24y^2 + 32y - 65 = 0$.

3. Transform the equation $x^3 - 7x^2 + 36 = 0$ into another whose second term shall be wanting.

Ans. $y^3 - \frac{49}{3}y + \frac{286}{27} = 0$.

EQUAL ROOTS.

470. If we consider that the roots of the general equation

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0 \quad (1)$$

are a, b, c, d, $\ldots\ldots$, we have (Art. 440) the identical equation

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots = (x-a)(x-b)(x-c)(x-d)\ldots\ldots$$

Substituting $x+y$ for x, we have

$$(x+y)^n + p(x+y)^{n-1} + \ldots\ldots = (x-a+y)(x-b+y)(x-c+y)\ldots\ldots$$

Developing each member of the equation, the coefficients of y must be equal, or,

$$\begin{aligned} n x^{n-1} + (n-1)\,p\,x^{n-2} + (n-2)\,q\,x^{n-3} + \ldots\ldots = \\ \left.\begin{array}{l} \quad (x-a)\,(x-b)\,(x-c)\ldots\ldots \\ +\,(x-a)\,(x-b)\,(x-d)\ldots\ldots \\ +\,(x-a)\,(x-c)\,(x-d)\ldots\ldots \\ +\,(x-b)\,(x-c)\,(x-d)\ldots\ldots \\ +\quad \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{to } n-1 \text{ factors.} \end{aligned}$$

Now, if $b = a$, it is evident that every term of the second member of this equation is divisible by $x - a$, and consequently its first member is divisible by the same factor; that is, if equation (1) has two roots equal to a, its first derived equation has one root equal to a. In the same manner, if $c = b = a$, that is, if (1) has

three equal roots, its first derived equation has two of them. In general,

If an equation has n *roots equal to* a, m *roots equal to* d, *etc., then its first derived equation must have* n — 1 *roots equal to* a, m — 1 *roots equal to* d, *etc.*

471. *To determine whether an equation has equal roots, and if so, to find them.*

From the principle demonstrated in the preceding Article, it is evident that we must

Seek the greatest common divisor of the first member of the given equation and its first derivative. If there is no common divisor, there can be no equal roots. If there is a common divisor, put that divisor equal to 0, *and the solution of this equation will give the required roots.*

The number of times that each of the equal roots occurs in the original equation is one greater than in the equation formed of the common divisor.

When the greatest common divisor is of a high degree, it should be compared with its own derivative, in order, if possible, to obtain a common divisor of a lower degree.

When the equal roots are known, it is evident that the given equation may be depressed to one containing only the remaining unequal roots (Art. 439), by dividing by the product of all the binomial factors corresponding to its equal roots

If the first member of the given equation be divided by the greatest common divisor, the resulting equation will contain all the different roots of the original equation, but without the repetition of any.

EXAMPLES.

Determine whether the following equations have equal roots, and obtain all their roots.

1. $x^4 - 14x^3 + 61x^2 - 84x + 36 = 0$.

Here, the first derived polynomial is $4x^3 - 42x^2 + 122x - 84$, and the greatest common divisor, obtained according to Art. 108, is $x^2 - 7x + 6$. The equation $x^2 - 7x + 6 = 0$ gives, according to the rules for quadratics, or by factoring, $x = 1$, or 6. Therefore, 1, 1, 6, and 6 are the roots of the given equation.

2. $x^3 - 8x^2 + 13x - 6 = 0$. Ans. 1, 1, and 6.

3. $x^3 - 7x^2 + 16x - 12 = 0$. Ans. 2, 2, and 3.

4. $x^4 - x^3 - 18x^2 + 52x - 40 = 0$. Ans. 2, 2, 2, and -5.

5. $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$.
Ans. 2, 2, 3, 3, and 3.

472. If an equation has two roots equal in magnitude, but opposite in sign, the signs of its alternate terms may be changed (Art. 457), and the common divisor of the two equations will contain those roots.

6. $x^4 + 3x^3 - 13x^2 - 27x + 36 = 0$.
Ans. 3, -3, 1, and -4.

SIGNS OF POLYNOMIAL FUNCTIONS.

473. A polynomial of the form

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v, \quad \text{(A)}$$

which we shall represent by X, may also be expressed thus (Art. 440):

$$(x - a)(x - b)(x - c)\ldots\ldots(x - l)\,Y, \quad \text{(B)}$$

in which $a, b, c, \ldots\ldots l$ are the real, unequal roots of the equation $X = 0$, in the order of their magnitude, a being *algebraically* the smallest, that is, nearest to $-\infty$. Y, the product of all the factors containing imaginary

roots, must always be positive, and cannot affect the sign of X, for each pair of imaginary roots (Art. 449) produces a factor of the form $x^2 + b^2$, or $(x - a)^2 + b^2$.

Suppose x to commence at $-\infty$, and proceed by insensible degrees through all possible negative values to 0, and thence through all possible positive values to $+\infty$.

When x has any value between $-\infty$ and a, each of the factors $(x - a)$, $(x - b)$, ... is negative, and therefore X is either positive or negative, according as the degree is even or odd (Art. 202).

When $x = a$, $X = 0$. When x is made greater than a and less than b, $x - a$ becomes positive, and the sign of X changes. When the value of x passes through b, X first becomes 0 and then changes sign, and so on, for each real root.

When x has any value between l and $+\infty$, X must be positive ; for all its factors are positive.

Hence, we learn that, if $X = 0$ has no equal roots,

Any function, as X, *varying by insensible degrees, always changes its sign when it passes through* 0, *and only then.*

The function X *experiences just as many changes of sign, when* x *varies from* $-\infty$ *to* $+\infty$, *as there are real roots of* $X = 0$.

Sturm's Theorem (Art. 476) enables us to determine the number of changes by the simple substitution of $-\infty$ and $+\infty$. If there are no changes of sign, the roots of $X = 0$ must all be imaginary, and X must be positive for all real values of x.

474. *If two numbers, when substituted for the unknown quantity in an equation, give results having a different sign, at least one root lies between those numbers.*

It is evident, from Art. 473, that if X has a different sign for two values of x, some *odd* number of roots lies between them.

When the numbers substituted differ by unity, it is evident that the integral part of the root is known.

EXAMPLES.

1. What is the first figure of a root of the equation $x^3 + 3x^2 - 7x - 8 = 0$?

Here, $x = 2$ gives $X = -2$, and $x = 3$ gives $X = 25$; hence 2 is the first figure of the root.

2. Find the integral parts of all the roots of the equation $x^3 - 6x^2 + 3x + 9 = 0$.

475. If any quantity *numerically greater* than either a or l (Art. 473) be substituted for x, each binomial factor of (B) will take the sign of x, and the sign of X will be the same as that of x^n.

If any quantity *numerically less* than each root of $X = 0$ be substituted for x, each binomial factor will take the sign of its last term, and the sign of X will be the same as that of v, its absolute term (Art. 441).

STURM'S THEOREM.

476. *To determine the number and situation of the real roots of any equation.*

A perfect solution of this difficult problem was first obtained by Sturm, in 1829. As the theorem which bears his name determines the number of real roots, the number of imaginary roots also becomes known (Art. 436).

477. Let X represent the first member of the equation

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v = 0,$$

from which the equal roots have been removed (Art. 471), and X_1 the first derived polynomial of X (Art. 467).

Divide X by X_1, and we shall obtain a quotient, Q_1, with a remainder of a lower degree than X_1. Represent this remainder, *with its signs changed*, by X_2, divide X_1 by X_2, and so on, as in finding the greatest common divisor (Art. 108), except that the signs of the remainders must be changed, while no other change of signs is admissible. As the equation $X = 0$ has no equal roots, there can be no common divisor, and the last remainder, X_n, will be independent of x. The successive operations may be represented by the following equations: —

$$\begin{aligned} X &= X_1 Q_1 - X_2 && (1) \\ X_1 &= X_2 Q_2 - X_3 && (2) \\ X_2 &= X_3 Q_3 - X_4 && (3) \\ &\cdots\cdots\cdots \\ X_{n-2} &= X_{n-1} Q_{n-1} - X_n && (n-1) \end{aligned}$$

478. Substitute any number for x in the series of functions represented by X, X_1, X_2, ... X_n, and write the signs of the results in a row. If any other number be substituted in like manner,

The difference between the number of variations in the first row of signs and that in the second, is equal to the number of real roots of the given equation comprised between the quantities substituted for x.

This is Sturm's Theorem, the demonstration of which depends upon the following principles.

1. *Two consecutive functions of the series* $X, X_1, X_2, \ldots\ldots X_n$ *cannot both become* 0 *for the same value of* x.

If $X_1 = 0$ and $X_2 = 0$, then, by equation (2), Art. 477, $X_3 = 0$; and if $X_2 = 0$ and $X_3 = 0$, then, by (3), $X_4 = 0$; and so on, till $X_n = 0$. But it is impossible that the last remainder shall become 0, for any value of x, because it is independent of x.

2. *When either function of the series, except* X *and* X_n,

becomes 0 *for a particular value of* x, *the two functions adjacent to it must have different signs.*

If $X_2 = 0$ when $x = b$, then neither X_1 nor X_3 can be 0 for that value of x, and, by equation (2), $X_1 = -X_3$; that is, X_1 and X_3 must have different signs.

It is evident that X_1 and X_3 will also continue to have different signs for values of x a little greater or less than b; for no change in the signs of X_1 or X_3 can take place until after the value of x is so much changed that one of them becomes 0 (Art. 473).

3. *When either function of the series, except* X *and* X_n, *changes its sign for different values of* x, *the* NUMBER *of variations is not affected by the change.*

If c and d, when substituted for x, give different signs for X_2, conceive all possible quantities between c and d to be substituted for x, in regular order, and X_2 changes its sign in passing through 0 (Art. 473). Hence, when X_2 changes its sign, as well as just before and just after the change, X_1 and X_3 must have different signs, and the sign of X_2, whether $+$ or $-$, must, therefore, in both cases, be like one of those adjacent to it, and unlike the other, giving, with X_1 and X_3, one permanence and one variation. The only effect of changing the sign is that the permanence and variation exchange places.

The last function, X_n, is independent of x, and therefore can never change its sign, whatever value may be given to x. Hence a change in the number of variations can be caused only by a change in the sign of the original function, X.

4. *When the original function,* X, *changes its sign for successive increasing values of* x, *the number of variations is diminished by one.*

Let a be a root of the original equation, or a value of x which causes X to become 0, and let A_1 represent the

value of X_1 when $x = a$. If we assume $x = a + y$ and $x = a - y$, y being too small for X_1 to become 0, then the sign of X_1 for each of these values of x will be the same as for $x = a$, that is, the same as the sign of A_1.

But if $a + y$ be substituted for x in X, we obtain (Art. 469)

$$A + A_1 y + \frac{A_2}{2} y^2 + \frac{A_3}{2 \cdot 3} y^3 + \ldots\ldots + y^n;$$

or, since A is merely a substituted for x in X, and therefore equal to 0,

$$y \left(A_1 + \frac{A_2}{2} y + \frac{A_3}{2 \cdot 3} y^2 + \ldots\ldots \right).$$

In the same manner, when $x = a - y$, X becomes

$$y \left(- A_1 + \frac{A_2}{2} y - \frac{A_3}{2 \cdot 3} y^2 + \ldots\ldots \right).$$

As these series are finite, y may be made so small that the sign of each shall depend upon that of A_1 (Art. 475). Hence, X has the same sign as A_1 when x is a little larger than a, and a different sign when it is a little smaller. But X_1 has the same sign as A_1. Hence, if x is increasing, X and X_1 have a different sign before X becomes 0, and the same sign after. A variation is therefore changed to a permanence.

The truth of the theorem is evident from the foregoing principles. If x take all possible values from p to q, p being less than q, one variation is changed to a permanence each time that X passes through 0 and changes its sign, and only then, for no change of sign in any of the other functions, X_1, X_2, etc., can affect the number of variations. Hence the number of variations lost in passing from p to q is exactly equal to the number of real roots of the equation $X = 0$ comprised between p and q.

479. When $-\infty$ and $+\infty$ are substituted for p and q, the whole number of real roots of the equation $X = 0$ becomes known (Art. 473).

The substitution of $-\infty$ and 0 will give the whole number of negative roots, while 0 and $+\infty$ will give the whole number of positive roots. When the roots are all real, Descartes' Rule (Art. 452) will answer the same purpose.

The substitution of various numbers for p and q will show between what numbers the roots lie, or fix the *limits* of the roots.

480. It is evident that X and X_1 must change signs alternately, as they are always unlike just before X changes. Hence, when the roots of $X = 0$ are all real, each root of $X_1 = 0$ must be intermediate in value between two roots of $X = 0$. For this reason the first derived equation is often called the limiting or separating equation (Art. 468).

481. In the process of finding X_2, X_3, etc., any *positive* numerical factors can be introduced or rejected at pleasure, since the *sign* of the result is not affected thereby. In this way fractions may be avoided.

In substituting $-\infty$ and $+\infty$, the first term of each function determines the sign (Art. 475).

Examples.

1. Determine the number and situation of the real roots of the equation $x^3 - 4x^2 - x + 4 = 0$.

Here, the first derived polynomial of the first member (Art. 467) is $3x^2 - 8x - 1$. Multiplying $x^3 - 4x^2 - x + 4$ by 3, to make its first term divisible by $3x^2$,

$$
\begin{array}{rl}
3x^2-8x-1) & 3x^3-12x^2-3x+12\ (x-2 \\
& \underline{3x^3-8x^2-x} \\
& -4x^2-2x+12 \\
& \qquad\qquad\qquad\quad \tfrac{3}{2} \\
& \underline{-6x^2-3x+18} \\
& -6x^2+16x+2 \\
& \underline{\qquad\qquad -19x+16} \quad \therefore\ X_2 = 19x-16
\end{array}
$$

$$
\begin{array}{rl}
19x-16) & 3x^2-8x-1\ (3x-104 \\
& \qquad\qquad\qquad\quad 19 \\
& \underline{57x^2-152x-19} \\
& 57x^2-48x \\
& \underline{\qquad -104x-19} \\
& \qquad\qquad\quad 19 \\
& \underline{\qquad -1976x-361} \\
& -1976x+1664 \\
& \underline{\qquad\qquad -2025} \quad \therefore\ X_3 = +2025.
\end{array}
$$

$$X = x^3-4x^2-x+4; \qquad X_2 = 19x-16;$$
$$X_1 = 3x^2-8x-1; \qquad X_3 = +2025.$$

The last step of the division may be omitted, for we can easily determine what the *sign* of X_3 must be, when $-104x-19$ is obtained.

We first substitute $-\infty$ for x in each function, and obtain three variations of sign; in like manner $+\infty$ gives no variation; hence the three roots are all real. By substituting 0, and comparing with the former results, or by applying Descartes' Rule, we learn that one root is negative and two are positive. We therefore substitute various numbers to determine the limits of the roots.

The following table presents all of these results in a connected form.

	X	X_1	X_2	X_3	
$x = -\infty$,	$-$	$+$	$-$	$+$,	3 variations.
$x = -2$,	$-$	$+$	$-$	$+$,	3 variations.
$x = -1$,	0	$+$	$-$	$+$	
$x = 0$,	$+$	$-$	$-$	$+$,	2 variations.
$x = 1$,	0	$-$	$+$	$+$	
$x = 2$,	$-$	$-$	$+$	$+$,	1 variation.
$x = 3$,	$-$	$+$	$+$	$+$,	1 variation.
$x = 4$,	0	$+$	$+$	$+$	
$x = 5$,	$+$	$+$	$+$	$+$,	no variation.
$x = \infty$,	$+$	$+$	$+$	$+$,	no variation.

If only -2 and 5 had been substituted, it would at once be seen that there must be three real roots between those numbers, for the former produces three more variations than the latter. But we also find that -1, 1, and 4 each reduce X to 0, and are, therefore, the three roots required.

NOTE. To obtain all the relations of signs, we might substitute -0.5 and 0.9 for x, the former giving $+ + - +$, 2 variations, and the latter $+ - + +$, 2 variations. These will serve to illustrate the demonstration of the third and fourth principles, Art. 478, although they are of no service in obtaining the roots of the given equation.

2. Determine the number and situation of the real roots of the equation $x^4 - 3x^3 + 3x^2 - 3x + \frac{5}{2} = 0$.

Here, $X = x^4 - 3x^3 + 3x^2 - 3x + \frac{5}{2}$;

$X_1 = 4x^3 - 9x^2 + 6x - 3$; $X_3 = -92x + 129$;

$X_2 = 3x^2 + 18x - 31$; $X_4 = -1163$.

	X	X_1	X_2	X_3	X_4	
$x = -\infty$,	$+$	$-$	$+$	$+$	$-$,	3 variations.
$x = 0$,	$+$	$-$	$-$	$+$	$-$,	3 variations.
$x = 1$,	$+$	$-$	$-$	$+$	$-$,	3 variations.
$x = 2$,	$+$	$+$	$+$	$-$	$-$,	1 variation.
$x = +\infty$,	$+$	$+$	$+$	$-$	$-$,	1 variation.

Substituting $-\infty$ and $+\infty$, we obtain a difference of two variations; hence there are two real and two imaginary roots. Substituting 0, we find the real roots to be positive. The substitution of 1 and 2 shows that both real roots lie between 1 and 2.

NOTE. X_1, X_2, and X_3 all *appear* to change signs at once, and affect the number of variations; but the substitution of numbers between 1 and 2 will show that X changes sign *twice* between those numbers, and the other functions conform to the principles already stated. Articles 473 and 474 give no evidence that two roots lie between 1 and 2, if the *signs* of X alone are considered, but Sturm's Theorem detects them at once.

3. Determine the number and situation of the real roots of the equation $x^3 - x^2 - 2x + 1 = 0$.

Ans. Three; one between 0 and 1, one between 1 and 2, and one between -1 and -2.

4. Determine the number and situation of the real roots of the equation $x^3 - 7x + 7 = 0$.

Ans. Three; two between 1 and 2, and one between -3 and -4.

5. Determine the number and situation of the real roots of the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$.

Ans. Four; between 0 and 1, 1 and 2, 2 and 3, -2 and -3.

6. Determine the number and situation of the real roots of the equation $x^3 - 2x - 5 = 0$.

Ans. One, between 2 and 3.

7. How many real roots has the equation $2x^4 - 3x^3 + 17x^2 - 3x + 15 = 0$? Ans. None.

8. Determine the number and situation of the real roots of the equation $x^4 - 4x^3 - 3x + 27 = 0$.

Ans. Two; one between 2 and 3, and one between 3 and 4.

SOLUTION OF HIGHER NUMERICAL EQUATIONS.

482. The real roots of the higher numerical equations must be obtained by tentative methods, or by methods which involve approximation. Equations of the third and fourth degrees may be considered as included in the class of higher equations ; for their general solutions are complicated, and only of limited application (Art. 508–514). No general solution of an equation higher than the fourth degree has yet been obtained.

If the proposed equation contains equal roots, they may be removed (Art. 471) ; and the student should also make use of any other principles found on the preceding pages, whenever the process can thereby be abbreviated.

COMMENSURABLE ROOTS.

483. Any equation containing fractional coefficients may be transformed into another whose coefficients shall be entire, that of the first term being unity (Art. 460), and such an equation cannot have a root equal to a rational fraction (Art. 445) ; hence, to find all rational or commensurable roots, we have only to find all integral roots.

484. Every rational root of an equation is a divisor of the last term (Art. 443); hence, to find the commensurable roots of an equation,

Ascertain by trial what integral divisors of the absolute term are roots of the equation.

The trial may be made by substituting each divisor, both with the positive and the negative sign, in the equation (Art. 431), or by dividing the first member of

the equation by the unknown quantity minus the supposed root (Art. 435). In substituting very small numbers, such as ± 1, the former method may be most convenient; but, when a true root has been used, the latter method will give at once the depressed equation, which may be used in obtaining the other roots (Art. 439).

485. When the number of divisors of the last term is large, this process of successive trials becomes tedious, and another method may be adopted.

If a is a root of the equation

$$x^4 + px^3 + qx^2 + tx + u = 0,$$

then

$$a^4 + pa^3 + qa^2 + ta + u = 0.$$

Transposing and dividing by a,

$$\frac{u}{a} = -t - qa - pa^2 - a^3,$$

and $\frac{u}{a}$ is a whole number.

In the same way, if we represent $\frac{u}{a} + t$ by t',

$$\frac{t'}{a} = -q - pa - a^2,$$

and $\frac{t'}{a}$ is a whole number.

Proceeding in this way, we find that when a is a root of the equation, it must satisfy the following conditions,

$$\frac{u}{a} + t = t', \qquad \frac{q'}{a} + p = p',$$

$$\frac{t'}{a} + q = q', \qquad \frac{p'}{a} + 1 = 0,$$

where each expression represents an entire quantity.

Hence the following

RULE.

Divide the absolute term of the equation by one of its integral divisors, and add to the quotient the coefficient of the first power of the unknown quantity.

Divide this sum by the same divisor, and, if the quotient is a whole number, add to it the coefficient of the second power of the unknown quantity.

Proceed in the same manner with each coefficient, in regular order, and, if the divisor is a root of the equation, each quotient will be entire, and the last quotient, added to the coefficient of the highest power of the unknown quantity, will be 0.

When the coefficient of the highest power is 1, the last quotient must be -1; and when any power is wanting, it must be supplied with the coefficient 0.

EXAMPLES.

1. Given $x^3 - 6x^2 + 27x - 38 = 0$, to find the values of x.

By Descartes' Rule (Art. 452) it is evident that the equation can have no negative root; and as the sum of the coefficients is not 0, it is also evident that 1 cannot be a root of the equation. The only remaining divisors of the last term are 2, 19, and 38. Dividing the first member by $x - 2$ (Art. 464),

$$\begin{array}{r|l} 1 - 6 + 27 - 38 & +2 \\ \quad + 2 - 8 + 38 & \\ \hline 1 - 4 + 19 \quad , \quad 0 & \end{array}$$

we obtain $x^2 - 4x + 19$, and no remainder; hence 2 is a root of the original equation, and $x^2 - 4x + 19 = 0$ is the depressed equation. Solving this by quadratics, we obtain $x = 2 \pm \sqrt{-15}$ for the other roots, and 2 is therefore the only rational root.

2. Given $8x^4 - 4x^3 - 14x^2 + x + 3 = 0$, to find the values of x.

Putting $x = \frac{y}{2}$, we obtain (Art. 460)

$$y^4 - y^3 - 7y^2 + y + 6 = 0. \qquad (1)$$

The divisors of 6 are ± 1, ± 2, ± 3, and ± 6.

By substituting $+1$ and -1 for y, it is readily seen that both are roots of the equation, and the other two roots can be found from the depressed equation. But all of the rational roots may be obtained from (1) by the method just explained.

Divisors,	$+6$,	$+3$,	$+2$,	$+1$,	-1,	-2,	-3,	-6
1st Quotients,	$+1$,	$+2$,	$+3$,	$+6$,	-6,	-3,	-2,	-1
Quo. $+1$,	$+2$,	$+3$,	$+4$,	$+7$,	-5,	-2,	-1,	0
2d Quotients,		$+1$,	$+2$,	$+7$,	$+5$,	$+1$,		0
Quo. -7,		-6,	-5,	0,	-2,	-6,		-7
3d Quotients,		-2,		0,	$+2$,	$+3$		
Quo. -1,		-3,		-1,	$+1$,	$+2$		
4th Quotients,		-1,		-1,	-1,	-1		
Quo. $+1$,		0,		0,	0,	0		

As 6, 2, -3, and -6 give fractional quotients, they cannot be roots of the equation, and are rejected. 3, 1, -1, and -2 give entire quotients, and the fourth quotient for each is -1; hence they are the four values of y, and $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and -1 must be the four values of x.

Find all the commensurable roots of the following equations, and also all of the roots, when possible, by methods already given.

3. $x^3 + 6x^2 + 11x + 6 = 0$. Ans. -1, -2, and -3.

4. $x^3 + 3x^2 - 4x - 12 = 0$. Ans. 2, -2, and -3.

5. $x^4 - 9x^3 + 23x^2 - 20x + 15 = 0$. Ans. 3.

6. $x^3 - 3x^2 + x + 2 = 0$. Ans. 2, 1.618, and -0.618.

7. $x^4 - 4x^3 - 8x + 32 = 0$.
Ans. 2, 4, and $-1 \pm \sqrt{-3}$.

8. $x^3 - 7x^2 + 36 = 0$. Ans. 3, 6, and -2.

9. $x^3 - 6x^2 + 10x - 8 = 0$. Ans. 4 and $1 \pm \sqrt{-1}$.

10. $x^4 + x^3 - 29x^2 - 9x + 180 = 0$.
Ans. 3, 4, -3, and -5.

11. $x^3 - 6x^2 + 11x - 6 = 0$.

12. $4x^3 - 16x^2 - 9x + 36 = 0$.
Ans. $\frac{3}{2}$, 4, and $-\frac{3}{2}$.

13. $2x^3 - 3x^2 + 16x - 24 = 0$.
Ans. $\frac{3}{2}$, and $\pm 2\sqrt{-2}$.

14. $x^5 - 2x^3 - 16 = 0$. Ans. 2.

INCOMMENSURABLE ROOTS.

486. If a higher numerical equation is found to contain no commensurable root, or if, after removing the commensurable roots, the depressed equation is still of a higher degree, the irrational or incommensurable roots must next be sought (Art. 446). The integral parts of these roots may be found by Sturm's Theorem, or by Art. 474, and the decimal parts by Arts. 515, 516, or by

HORNER'S METHOD.

487. Suppose that a root of the equation

$$x^n + px^{n-1} + qx^{n-2} + \ldots + tx^2 + ux + v = 0 \quad (1)$$

is found to lie between a and $a + 1$. Transform the equation into another whose roots shall be less by a (Art. 462), and we shall have one of the form

$$y^n + p'y^{n-1} + q'y^{n-2} + \ldots + t'y^2 + u'x + v' = 0, \quad (2)$$

one of whose roots is less than 1. If that root is found to be between the decimal fraction a' and $a' + .1$, transform equation (2) into another whose roots shall be less by a', and we shall have one of the form

$$z^n + p''z^{n-1} + q''z^{n-2} + \ldots\ldots + t''x^2 + u''z + v'' = 0, \quad (3)$$

one of whose roots is less than .1. If that root is found to be a little greater than a'', proceed with (3) in the same manner as before, and we obtain

$$x = a + a' + a'' + \ldots\ldots$$

to any required degree of accuracy.

As z, in equation (3), represents a small fraction, its higher powers will be so small that they may be neglected, and its value is nearly $-\frac{v''}{u''}$; therefore a', a'', ... may be found in this way, with more and more accuracy the smaller they become.

Hence, a positive incommensurable root of an equation may be obtained approximately by the following

RULE.

Find by Sturm's Theorem, or by trial, the initial part of the root, and transform the given equation into another whose roots shall be less by that initial part.

Divide the absolute term of the transformed equation by the coefficient of the first power of its unknown quantity, for the next figure of the root.

Transform this last equation into another whose roots shall be less by the figure of the root last found, divide as before for the next figure of the root, and so on.

488. A negative root may be found by changing the signs of the alternate terms of the equation; and then treating the root as a positive one (Art. 457).

489. The *first* division will not always give the correct figure. If the wrong figure is used, the mistake will become evident in obtaining the next figure, and the correct one may be obtained by trial, or by Sturm's theorem. When the equation has two roots whose initial parts are the same, it is best to determine the next figures by Sturm's theorem, or by Art. 474.

490. The last two terms of the transformed equation should have *opposite* signs, in order that the quotient should be the proper figure to *add* to the part of the root already found, for it has been shown that $a'' = -\frac{v''}{u''}$ (Art. 487).

EXAMPLES.

1. Solve the equation $x^3 - x^2 - 2x + 1 = 0$.

By Sturm's theorem we learn that the roots of this equation are all real, two being positive and one negative, situated between 1 and 2, 0 and 1, and -1 and -2.

To find the first root, we substitute $y + 1$ for x. This transformation may be effected by either of the methods given in Arts. 462–465, but the method by synthetic division is to be preferred.

$$
\begin{array}{llll|l}
1 & -1 & -2 & +1 & +1 \\
 & +1 & 0 & -2 & \\
\hline
 & 0 & -2 & -1 & \\
 & +1 & +1 & & \\
\hline
 & +1 & -1 & & \\
 & +1 & & & \\
\hline
 & +2 & & &
\end{array}
$$

The transformed equation, whose root is less than 1, is then $y^3 + 2y^2 - y - 1 = 0$.

The second paragraph of the rule will not apply to this

equation (Arts. 489, 490). We try .9, and find it too large, and then take .8, as follows,

1	+2	−1	−1	.8
	.8	2.24	.992	
	2.8	1.24	−.008	
	.8	2.88		
	3.6	4.12		
	.8			
	4.4			

The transformed equation, whose root is less than .1, is then $z^3 + 4.4z^2 + 4.12z - .008 = 0$.

Dividing .008 by 4.12 gives us in this case .002, nearly, the next two figures of the root. Hence, $x = 1.802$.

The other positive root is itself less than 1. Its first approximate value, .5, is found by trial to be too large, and we therefore take .4 as the first figure. Carrying this out according to the rule, we obtain .445, nearly.

Changing the signs of the alternate terms of the original equation (Art. 488), its root is found to be 1.247.

Hence the three roots of the original equation are 1.802, .445, and − 1.247.

2. Required the real roots of the equation

$$x^4 - 4x^3 - 3x + 27 = 0.$$

The successive transformations are usually written in connection, as in the following form, where the different coefficients are indicated by the *. The work may also be contracted by dropping such decimal figures from the right of each column as are not needed for the required degree of accuracy.

By Sturm's theorem, or by Arts. 453, 474, the equation is found to have two real and two imaginary roots, the former being situated between 2 and 3, and 3 and 4.

1	-4	± 0	-3	$+27$	2.26745
	2	-4	-8	-22	
	-2	-4	-11	$*5.0000$	
	2	0	-8	-3.7664	
	0	-4	$*-19.000$	$*1.2336$	
	2	$+4$	.168	-1.0987	
	2	$*0.00$	-18.832	$*.1349$	
	2	.84	.344	$-.1267$	
	$*4.0$	.84	$*-18.488$	$*.0082$	
	.2	.88	.176	$-.0072$	
	4.2	1.72	-18.312	$*.0010$	
	.2	.92	.194	$-.0009$	
	4.4	$*2.64$	$*-18.118$	.0001	
	.2	.29	.025		
	4.6	2.93	-18.093		
	.2	.30	.025		
	$*4.80$	3.23	$*-18.07$		
	.06	.30			
	4.86	$*3.5$			
	.06				
	5.				

In like manner the other root is found to be 3.6796.

Obtain the real roots of the four following equations.

3. $x^3 - 2x - 5 = 0$. Ans. 2.09455.

4. $x^3 + x^2 - 500 = 0$. Ans. 7.61728.

5. $x^3 - 7x + 7 = 0$.

Ans. 1.3569, 1.6920, and -3.0489.

6. $x^3 - 17x^2 + 54x - 350 = 0$. Ans. 14.95407.

7. Obtain one root of the equation $x^4 - 3x^2 + 75x - 10000 = 0$. Ans. 9.88600.

APPENDIX.

RECURRING OR RECIPROCAL EQUATIONS.

491. A RECURRING EQUATION is one in which the coefficients of any two terms equally distant from the extremes of the first member are equal.

The equal coefficients may have the same sign, or opposite signs; but a part cannot have the same and a part opposite signs, in the same equation. Also, if the degree be even, and the equal coefficients have opposite signs, the middle term must be wanting.

492. *If any quantity is a root of a recurring equation, the reciprocal of that quantity is also a root of the same equation.*

Let $x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots \pm (\ldots\ldots + q\,x^2 + p\,x + 1) = 0 \quad (1)$

be the equation. If it be transformed into another whose roots are the reciprocals of those of the given equation (Art. 461), the transformed equation will coincide with the given equation, after changing signs throughout, if necessary. The two equations must therefore have the same roots. If a is a root of the given equation, $\frac{1}{a}$ is a root of the transformed equation, and therefore $\frac{1}{a}$ is a root of the given equation. In like manner, if b is a root of any recurring equation, $\frac{1}{b}$ is a root of the same equation, and so on.

493. On account of the property just demonstrated, *recurring* equations are also called *reciprocal* equations, the former term relating to their *coefficients*, and the latter to their *roots*.

494. *One root of a recurring equation of an odd degree is* -1 *when the equal coefficients have the same sign, and* $+1$ *when they have opposite signs.*

A recurring equation of an odd degree, as

$$x^{2m+1} + p x^{2m} + q x^{2m-1} + \ldots\ldots \pm (\ldots\ldots q x^2 + p x + 1) = 0, \quad (2)$$

has an even number of terms, and may be written in one of the following forms,

$$(x^{2m+1} + 1) + p\,(x^{2m} + x) + q\,(x^{2m-1} + x^2) + \ldots\ldots = 0$$
$$(x^{2m+1} - 1) + p\,(x^{2m} - x) + q\,(x^{2m-1} - x^2) + \ldots\ldots = 0.$$

If -1 be substituted for x in the first form, or $+1$ in the second, the first member will become 0; hence -1 is a root of the first and $+1$ a root of the second.

If equation (2) be divided by $x \pm 1$, both forms will reduce to the following form,

$$x^{2m} + p x^{2m-1} + q x^{2m-2} + \ldots\ldots + q x^2 + p x + 1 = 0, \quad (3)$$

a recurring equation of an even degree, in which the equal coefficients have the same sign. Hence, a recurring equation of an odd degree may always be depressed to one of an even degree.

495. *Two roots of a recurring equation of an even degree are* $+1$ *and* -1 *when the equal coefficients have opposite signs.*

As the middle term must be wanting (Art. 491), such an equation may be written in the form

$$(x^{2m} - 1) + p x\,(x^{2m-2} - 1) + q x^2\,(x^{2m-4} - 1) + \ldots\ldots = 0, \quad (4)$$

which is evidently divisible by both $x - 1$ and $x + 1$, or by $x^2 - 1$ (Art. 99). Hence, $+1$ and -1 are both roots of the equation.

If equation (4) be divided by $x^2 - 1$, it will be depressed two degrees, and become a recurring equation of an even degree, in which the equal coefficients have the same sign. Hence, every recurring equation may be depressed to the form of equation (3).

496. *Every recurring equation of an even degree, whose equal coefficients have the same sign, may be reduced to an equation of half that degree.*

Divide the equation

$$x^{2m} + p x^{2m-1} + q x^{2m-2} + \ldots\ldots + q x^2 + p x + 1 = 0$$

by x^m, and it may be written thus,

$$\left(x^m+\frac{1}{x^m}\right)+p\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+q\left(x^{m-2}+\frac{1}{x^{m-2}}\right)+\ldots=0 \quad (5)$$

the middle term, if present, becoming a known quantity.

Put

$$x+\frac{1}{x}=y$$

$$x^2+\frac{1}{x^2}=y^2-2$$

$$x^3+\frac{1}{x^3}=y^3-3\left(x+\frac{1}{x}\right)=y^3-3y$$

$$x^4+\frac{1}{x^4}=(y^2-2)^2-2=y^4-4y^2+2$$

. . . . . .

$$x^m+\frac{1}{x^m}=y^m-my^{m-2}+\ldots\ldots$$

Substituting these values in equation (5), we have an equation of the form

$$y^m+p'y^{m-1}+q'y^{m-2}+\ldots\ldots=0.$$

After this equation is solved, we can immediately find x from the equation $x+\frac{1}{x}=y$.

497. It thus appears that any recurring equation of the $(2m+1)$th degree, one of the $(2m+2)$th degree whose equal coefficients have opposite signs, and one of the $2m$th degree whose equal coefficients have the same sign, may each be reduced to an equation of the mth degree.

EXAMPLES.

1. Given $x^4-5x^3+6x^2-5x+1=0$, to find x.

Dividing by x^2, $\quad x^2-5x+6-\frac{5}{x}+\frac{1}{x^2}=0$

Or, $\quad \left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0$

Substituting y for $x+\frac{1}{x}$, and y^2-2 for $x^2+\frac{1}{x^2}$,

$$y^2-5y+4=0$$

Whence, $\quad y=4$, or 1

$$x+\frac{1}{x}=4, \text{ gives } x=2\pm\sqrt{3}$$

$$x + \frac{1}{x} = 1, \text{ gives } x = \tfrac{1}{2}(1 \pm \sqrt{-3}).$$

That $2 - \sqrt{3}$ and $\frac{1 - \sqrt{-3}}{2}$ are reciprocals of $2 + \sqrt{3}$ and $\frac{1 + \sqrt{-3}}{2}$ may easily be shown by rationalizing the denominators of $\frac{1}{2 + \sqrt{3}}$ and $\frac{2}{1 + \sqrt{-3}}$ (Art. 250).

2. Find the roots of the equation $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0$. Ans. -1, $\frac{9 \pm \sqrt{77}}{2}$, and $\frac{3 \pm \sqrt{5}}{2}$.

3. Given $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$, to find its roots.

Ans. -1, 1, 1, and $\frac{-3 \pm \sqrt{5}}{2}$.

4. Given $x^5 - x^5 + x^4 - x^2 + x - 1 = 0$, to find its roots.

Ans. -1, 1, $\pm\sqrt{-1}$, and $\frac{1 \pm \sqrt{-3}}{2}$.

5. Given $x^3 + px^2 + px + 1 = 0$, to find its roots.

Ans. -1 and $\frac{1}{2}(1 - p \pm \sqrt{p^2 - 2p - 3})$.

6. Given $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$, to find its roots.

Ans. 2, $\frac{1}{2}$, -3, and $-\frac{1}{3}$.

7. Given $5x^5 - 51x^4 + 160x^3 - 160x^2 + 51x - 5 = 0$, to find its roots. Ans. 1, 5, $\frac{1}{5}$, and $2 \pm \sqrt{3}$.

8. Given $x^4 + 5x^3 + 5x + 1 = 0$, to find its roots.

Ans. $x = -\frac{5}{4} \pm \frac{1}{4}\sqrt{33} \pm \frac{1}{4}(42 \mp 10\sqrt{33})^{\frac{1}{2}}$.

BINOMIAL EQUATIONS.

498. A BINOMIAL EQUATION is one which can be reduced to the form

$$x^n \pm v = 0 \qquad (1)$$

Some of the roots of binomial equations have already been obtained; for *pure equations* are binomial (Art. 269). We propose to state only a few general properties, and obtain the remaining roots of binomial equations of the lower degrees, since binomial equations

in general are most readily solved by means of trigonometrical formulas.

499. *The roots of a binomial equation are all unequal.*

For, $x^n \pm v$ and $n x^{n-1}$, its derived function, evidently have no common divisor (Art. 471).

500. *Every binomial equation may be transformed to a recurring equation.*

Put $x = v^{\frac{1}{n}} y$, or $x^n = v y^n$, and (1) becomes, after dividing by v,

$$y^n \pm 1 = 0, \qquad (2)$$

which is a recurring equation whose middle terms are wanting.

It is evident that if the roots of (2) are found, those of (1) become known; hence, (2) is used in place of (1).

501. If $2m+1$ and $2m$ represent n when odd and even, respectively, then (Art. 453)

$y^{2m+1} + 1 = 0$ has only one real root, -1.
$y^{2m+1} - 1 = 0$ has only one real root, $+1$.
$y^{2m} + 1 = 0$ has no real root.
$y^{2m} - 1 = 0$ has two real roots, $+1$ and -1.

502. *If* a *be one of the imaginary roots of the equation* $x^n - 1 = 0$, *then any power of* a *is also a root.*

For, since $a^n = 1$, $a^{2n} = 1$, $a^{3n} = 1$, etc.; also, $a^{-n} = 1$, $a^{-2n} = 1$, etc. Therefore, $\ldots\ldots a^{-2}, a^{-1}, 1, a, a^2, a^3, \ldots\ldots$ are all roots of the equation.

503. *If* a *be one of the imaginary roots of the equation* $x^n + 1 = 0$, *then any odd power of* a *is also a root.*

For, since $a^n = -1$, $a^{2n} = -1$, $a^{5n} = -1$, etc.; also, $a^{-n} = -1$, $a^{-3n} = -1$, etc. Therefore, $\ldots\ldots a^{-3}, a^{-1}, a, a^3, a^5, \ldots\ldots$ are all roots of the equation.

504. Although the roots of $x^n \pm 1 = 0$ may take an infinite variety of forms, as shown in the last two Articles, yet all these different forms must be capable of reduction to n values essentially different (Art. 436).

505. When n is composed of two or more integral factors, as in the equation $x^{pq} \pm 1 = 0$, let $x^p = y$, and we have $y^q \pm 1 = 0$.

Having found the q values of y, put x^p equal to each of them, find the p roots of each equation, and we have the pq values of x.

When n contains no other prime factors than 2, 3, and 5, the roots of a binomial equation may always be obtained by means of quadratics.

506. Since the n roots of equation (1) are expressed by $\sqrt[n]{\mp v}$, it follows that any quantity has as many roots of a given degree as there are units in the index of that degree. No more than two of these roots can be real, however.

EXAMPLES.

1. What are the roots of the equation $x^4 + 1 = 0$?

Dividing by x^2, $\quad x^2 + \frac{1}{x^2} = 0$

Substituting y for $x + \frac{1}{x}$, $\quad y^2 - 2 = 0$

Whence, $\quad y = \pm\sqrt{2}$

Replacing $x + \frac{1}{x}$, $\quad x^2 \mp x\sqrt{2} = -1$

Therefore, $\quad x = \pm\sqrt[4]{-1} = \pm\tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}\sqrt{-2}$.

2. What are the five fifth roots of 32?

The equation $x^5 = 32$ may be transformed into $y^5 - 1 = 0$, by substituting $2y$ for x. One root of this equation is evidently 1. Dividing $y^5 - 1$ by $y - 1$, we obtain

$$y^4 + y^3 + y^2 + y + 1 = 0.$$

This equation may be solved in the same manner as the last, by substituting z for $y + \frac{1}{y}$, or as follows,

$$(y^2 + \tfrac{1}{2}y)^2 + 2(y^2 + \tfrac{1}{2}y) + 1 = \tfrac{5}{4}y^2$$

$$y^2 + \tfrac{1}{2}y + 1 = \pm y\tfrac{1}{2}\sqrt{5}$$

$$y^2 + \tfrac{1}{2}(1 \mp \sqrt{5})\,y = -1$$

$$y = \tfrac{1}{2}(-1 \pm \sqrt{5} \pm \sqrt{-10 \mp 2\sqrt{5}})$$

$x = 2$, or $\tfrac{1}{2}(-1 + \sqrt{5} + \sqrt{-10 - 2\sqrt{5}})$,

or $\tfrac{1}{2}(-1 + \sqrt{5} - \sqrt{-10 - 2\sqrt{5}})$,

or $\tfrac{1}{2}(-1 - \sqrt{5} + \sqrt{-10 + 2\sqrt{5}})$,

or $\tfrac{1}{2}(-1 - \sqrt{5} - \sqrt{-10 + 2\sqrt{5}})$,

3. What are the three cube roots of 1?

Ans. $1, \frac{1}{2}(-1+\sqrt{-3}), \text{ and } \frac{1}{2}(-1-\sqrt{-3})$.

NOTE. In this case, either imaginary root is the square of the other, and the real root is the cube of either imaginary root, or the product of both.

4. Obtain the three roots of $x^3 + a^3 = 0$.

Ans. $-a, \frac{a}{2}(1+\sqrt{-3}), \text{ and } \frac{a}{2}(1-\sqrt{-3})$.

5. What are the four fourth roots of 1?

Ans. $1, -1, +\sqrt{-1}, \text{ and } -\sqrt{-1}$.

6. Obtain the roots of $x^5 + 1 = 0$.

Ans. $-1, \text{ and } \frac{1}{4}(1 \pm \sqrt{5} \pm \sqrt{-10 \pm 2\sqrt{5}})$,

in which $\sqrt{5}$ and $2\sqrt{5}$ must have the same sign.

7. Obtain the roots of $x^6 - a^6 = 0$.

Ans. $\pm a, \text{ and } \pm \frac{a}{2}(1 \pm \sqrt{-3})$.

507. When n is a fraction in its lowest terms, the numerator indicates the number of roots, and the denominator shows how the use of those roots is restricted. Thus, in the equation

$$x^{\frac{r}{s}} \pm 1 = 0,$$

x has r values, but each of these values is restricted to one of its s roots.

8. Obtain the roots of $x^{\frac{3}{2}} - 8 = 0$.

Ans. $4, \text{ and } 2(-1 \pm \sqrt{-3})$;

but each of these three values is restricted to its positive square root; that is, it can be used in only one of its two senses.

9. Obtain the roots of $x^{\frac{4}{3}} - 81 = 0$.

Ans. $\pm 27, \text{ and } \pm 27\sqrt{-1}$;

but each of these four values is restricted to one of its three cube roots.

CARDAN'S RULE FOR CUBICS.

508. In order to solve the general cubic equation,

$$x^3 + ax^2 + bx + c = 0,$$

it must first be transformed into another equation whose second term shall be wanting (Art. 466). Substituting $y - \frac{a}{3}$ for x, the equation takes the form

$$y^3 + py + q = 0.$$

Put $y = z - \frac{p}{3z}$, and the equation becomes

$$z^3 - \frac{p^3}{27z^3} + q = 0, \text{ or } z^6 + qz^3 = \frac{p^3}{27}$$

Solving this as a quadratic,

$$z^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \text{ and } z = \left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}$$

$$y = z - \frac{p}{3z} = \left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} - \frac{p}{3\left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}} \quad (A)$$

$$y = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} \quad (B)$$

(A) and (B) are different forms of Cardan's formula for the solution of cubic equations. In the former, both double signs are alike, that is, either both + or both —. (B) is derived from (A), by rationalizing the denominator (Art. 250), that is, multiplying both numerator and denominator of the fraction by $\left(-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}$. The double signs have also been dropped. After y is known, x is easily found.

From the above operation it will be seen that any cubic equation may be made to take a quadratic form, by the following process.

RULE.

Transform the given equation into another in which the second power of the unknown quantity shall be wanting (Art. 466).

Substitute for the unknown quantity a new unknown quantity, together with its reciprocal multiplied by one third of the coefficient of the second term of the equation with its sign changed.

Irreducible Case.

509. When p is negative, and $\frac{p^3}{27} > \frac{q^2}{4}$, Cardan's formula contains imaginary expressions, and yet it is easily shown that the three roots of the equation are then real and unequal.

The result of Cardan's solution contains imaginary expressions, unless two of the roots are imaginary or equal.

The three roots of the equation $y^3 + p\,y + q = 0$ may be represented by $a + \sqrt{3\,b}$, $a - \sqrt{3\,b}$, and $-2\,a$, where a and b may be either positive or negative, and b may be 0 (Arts. 442, 447–450). Then, by Art. 441,

$$p = a^2 - 3\,b - 2a^2 - 2a\sqrt{3b} - 2a^2 + 2a\sqrt{3b} = -3a^2 - 3b$$

$$q = 2\,a\,(a + \sqrt{3\,b})\,(a - \sqrt{3\,b}) = 2\,a\,(a^2 - 3\,b)$$

$$\left(\frac{p}{3}\right)^3 = -(a^2 + b)^3 = -a^6 - 3\,a^4 b - 3\,a^2 b^2 - b^3$$

$$\left(\frac{q}{2}\right)^2 = a^2\,(a^2 - 3\,b)^2 = a^6 - 6\,a^4 b + 9\,a^2 b^2$$

$$\frac{q^2}{4} + \frac{p^3}{27} = a^2\,(a^2 - 3\,b)^2 - (a^2 + b)^3 = -9\,a^4 b + 6\,a^2 b^2 - b^3$$

$$\sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = (3\,a^2 - b)\,\sqrt{-b}.$$

Therefore, when the roots are real and unequal, b is positive, and Cardan's formula must contain imaginary expressions; but when two of the roots are imaginary, b is negative, and the radical of the second degree in Cardan's formula becomes real. When two of the roots are equal, $b = 0$, and the radicals of the second degree disappear from both the roots and the formula.

510. Although the formula is irreducible by the common rules when the three roots are real and unequal, yet there are means by which it may be reduced, and shown to be real. When the cube root of the binomial surds is extracted, by developing into a series or otherwise, the imaginary expressions will have different signs, and cancel each other. Thus, the equation $x^3 - 6\,x + 4 = 0$ gives $x = (-2 + 2\sqrt{-1})^{\frac{1}{3}} + (-2 - 2\sqrt{-1})^{\frac{1}{3}}$, which is proved by trial to be the same as $1 + \sqrt{-1} + 1 - \sqrt{-1} = 2$. Also, $x^3 - 30\,x - 36 = 0$ gives $x = (18 + 26\sqrt{-1})^{\frac{1}{3}} + (18 - 26\sqrt{-1})^{\frac{1}{3}} = 3 + \sqrt{-1} + 3 - \sqrt{-1} = 6$. We have no *general* rule for

the extraction of the cube root of a binomial surd; but it may always be developed into a series by the binomial theorem.

The table of sines and tangents may also be employed in solving both the reducible and irreducible cases.

511. As any quantity has three cube roots (Art. 506), z evidently has six values (Art. 436); but the use of the double sign in (B) would not increase the number of values of y, since one sign must be the opposite of the other. y has only three values, one derived from the common cube root of each expression, and two derived from the other cube roots. After the real root is found by means of the common cube root, the other two are readily found by solving the quadratic obtained by depressing the equation (Art. 444).

512. It will be seen that this formula may always be used when p is positive (Art. 453). It can also be employed when p is negative, provided that $27q^2$ is numerically greater than $4p^3$.

513. The following examples can be solved by the method pointed out in the rule (Art. 508), or by a direct substitution in formulas (A) or (B), after removing the term containing x^2.

EXAMPLES.

1. Obtain the roots of the equation $x^3 - 3x^2 + 4 = 0$.

Substituting $y + 1$ for x, $\quad y^3 - 3y + 2 = 0$

Substituting $z + \frac{1}{z}$ for y, $\quad z^3 + \frac{1}{z^3} + 2 = 0$

Or, $\quad z^6 + 2z^3 + 1 = 0$

Whence, $\quad z^3 + 1 = 0$

$$z = -1, \text{ or } \frac{1 \pm \sqrt{-3}}{2}$$

$$y = z + \frac{1}{z} = -2, \text{ or } \frac{1 \pm \sqrt{-3}}{2} + \frac{1 \mp \sqrt{-3}}{2} = -2, \text{ or } 1, \text{ or } 1.$$

$$x = y + 1 = -1, \text{ or } 2, \text{ or } 2.$$

The value of y may also be obtained by substituting -3 for p, and 2 for q, in either (A) or (B), when $\sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ becomes 0, and disappears.

After $x = -1$ is obtained, the other values of x might be found by dividing $x^3 - 3x^2 + 4$ by $x + 1$, and solving the equation $x^2 - 4x + 4 = 0$.

2. Solve the equation $x^3 - 9x + 28 = 0$.

Substituting $z + \frac{3}{z}$ for x, $\quad z^6 + 28z^3 + 27 = 0$

Solving by quadratics, $\quad z^3 = -1$, or -27

Whence, $\quad z = -1$, or -3

$$x = z + \frac{3}{z} = -1 + \frac{3}{-1}, \text{ or } -3 + \frac{3}{-3} = -4.$$

Dividing $x^3 - 9x + 28$ by $x + 4$, and solving the equation $x^2 - 4x + 7 = 0$, the other roots are found to be $2 \pm \sqrt{-3}$.

$x = -4$ may also be obtained by substituting directly in either (A) or (B).

3. Obtain one root of the equation $x^3 + 6x - 2 = 0$.

Ans. $\sqrt[3]{4} - \sqrt[3]{2}$.

Obtain the roots of the following equations by Cardan's method.

4. $x^3 - 6x + 9 = 0$. Ans. -3, or $\frac{3 \pm \sqrt{-3}}{2}$.

5. $x^3 - 6x^2 + 57x - 196 = 0$. Ans. 4, or $1 \pm 4\sqrt{-3}$.

6. $x^3 - 4x^2 - 3x + 18 = 0$. Ans. 3, 3, or -2.

7. $x^3 - 24x^2 - 24x - 25 = 0$. Ans. 25, or $\frac{-1 \pm \sqrt{-3}}{2}$.

8. $x^3 + 9x^2 - 21x + 11 = 0$. Ans. 1, 1, or -11.

9. $x^3 - 2x^2 + 2x - 1 = 0$. Ans. 1, or $\frac{1 \pm \sqrt{-3}}{2}$.

10. $x^3 + 6x - 20 = 0$. Ans. 2, or $-1 \pm 3\sqrt{-1}$.

NOTE. If $\frac{q^2}{4} + \frac{p^3}{27}$ is positive, but not a perfect square, it does not always follow that the real root of the equation is irrational. The square and cube roots may be obtained approximately by means of decimals, and the decimals in the two parts of the formula may destroy each other, as imaginary quantities always do in the irreducible case. Sometimes the cube root of the binomial surd may be extracted. Thus,

$$(10 + 6\sqrt{3})^{\frac{1}{3}} + (10 - 6\sqrt{3})^{\frac{1}{3}} = 1 + \sqrt{3} + 1 - \sqrt{3} = 2.$$

BIQUADRATIC EQUATIONS.

514. General solutions of biquadratic equations have been obtained by Descartes, Simpson, Euler, and others. Some of them require the second term of the equation to be removed, while others do not. All of them depend upon the solution of a cubic equation by Cardan's rule, and will of course fail when that fails (Art. 509). In general, if an equation has no equal roots, these methods cannot be applied without the inconvenience of the irreducible case, unless two of the roots are real and two imaginary. Hence they are practically of little value (Art. 482), especially as numerical equations of all degrees can readily be solved by Horner's method (Art. 487), or by one of the following methods of approximation.

APPROXIMATION BY DOUBLE POSITION.

515. Find two numbers, a and b, the one greater and the other less than a root of the equation (Art. 474), and suppose a to be nearer the root than b. Substitute them separately for x in the given equation, and let A and B represent the values of the first member thus obtained. If a and b were true roots, A and B would each be 0; hence the latter may be considered the errors which result from substituting a and b for x. Although not strictly correct, yet, for the purpose of approximation, we may assume that

$$A : B :: x - a : x - b,$$

whence (Art. 316), $A - B : A :: b - a : x - a,$

or (Art. 313), $A - B : b - a :: A : x - a,$

and $$x - a = \frac{A\,(b-a)}{A-B},$$

or $$x = a + \frac{A\,(b-a)}{A-B}.$$

That is,

As the difference of the errors is to the difference of the two assumed numbers, so is either error to the correction of its assumed number.

Adding this correction when its assumed number is too small, or subtracting when too large, we obtain a nearer approximation to the

true root. This result and another assumed number may now be used as new values of a and b, for obtaining a still nearer approximation, and so on.

It is best to employ two assumed quantities that shall differ from each other only by unity in the last figure on the right. It is also best to use the smaller error.

This method of approximation has the advantage of being applicable to equations in any form. It may, therefore, be applied to radical and exponential equations (Art. 421), and others not reduced to the general form (Art. 429).

Examples.

1. Find one root of the equation $x^3 + x^2 + x - 100 = 0$.

When 4 and 5 are substituted for x in the equation, the results are -16 and $+55$, respectively; hence $a = 4$, $b = 5$, $A = -16$, and $B = 55$. According to the above formula, the first approximation gives $x = 4 + \frac{-16}{-71} = 4.2 +$

As the true root is greater than 4.2, we now assume 4.2 and 4.3. Substituting these values for x in the original equation, we obtain -4.072 and $+2.297$; therefore, 4.3 is nearer the true root than 4.2.

Hence, $x = 4.3 - \frac{.2297}{6.369} = 4.3 - .036 = 4.264.$

Substituting 4.264 and 4.265 for x, and stating the result in the form of a proportion, we have

$$.0276 + .0365 : .001 :: .0276 : .00043.$$

Hence, $x = 4.264 + .00043 = 4.26443.$

Find one root of each of the following equations.

2. $x^3 - 2x - 50 = 0.$ Ans. 3.864854.

3. $x^3 + 10x^2 + 5x - 260 = 0.$ Ans. 4.11799.

4. $x^3 + 8x^2 + 6x - 75.9 = 0.$ Ans. 2.4257.

5. $x^3 + \frac{11}{16}x - \frac{3}{4} = 0.$ Ans. .66437.

6. $x^4 - 3x^2 - 75x - 10000 = 0.$ Ans. 10.2609.

7. $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0.$ Ans. 8.414455.

NEWTON'S METHOD OF APPROXIMATION.

516. Find two numbers, one greater and the other less than a root of the equation (Art. 474). Let a be one of those numbers, the nearest to the root, if it can be ascertained. Substitute $a + y$ for x in the given equation; then y is small, and by omitting $y^2, y^3, \ldots\ldots$, as in Horner's method (Art. 487), a value of y is obtained, which, added to a, gives b, a closer approximation to the value of x. Now substitute $b + z$ for x in the given equation, and a second approximation may be obtained by the same process as before. By proceeding in this way, the value of the root may be obtained to any required degree of accuracy.

The assumed value of x should be nearer to one root than to any other, in order to secure accuracy in the approximation.

Examples.

1. Find the real root of the equation $x^3 - 2x - 5 = 0$.

When 2 and 3 are substituted for x in the equation, the results are -1 and $+16$, respectively; hence a root lies between 2 and 3, and near to 2. Substitute $2 + y$ for x, and there results

$$1 - 10y - 6y^2 - y^3 = 0,$$

whence, approximately, $y = .1$.

Now substitute $2.1 + z$ for x, and there results

$$.061 + 11.23z + \ldots\ldots = 0,$$

whence, approximately, $z = -\frac{.061}{11.23} = -.0054$, and

$$x = 2.1 - .0054 = 2.0946, \text{ nearly.}$$

Find one root of each of the following equations.

2. $x^3 - 3x + 1 = 0$. Ans. 1.53209.

3. $x^3 - 15x^2 + 63x - 50 = 0$. Ans. 1.02804.

THE END.

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